§1 Motivation in Studying Fluid Mechanics

At its core, fluid mechanics studies fluids in motion. Despite the simple definition, modeling fluids in the real world is complicated and arguably one of the hardest fields to study due to the difficulty in predicting patterns. Think of forecasting weather as an example: we usually forecast temperature and conditions 10 days in advance because it is too difficult to predict spontaneous weather patterns! Even then, forecasts are not always correct either. Thus, weather is calculated stochastically−meteorologists measure the chance of a climate condition occurring at a time and place. Despite the unpredictability of fluids, we cannot ignore their invaluable applications.

- *Monitoring blood flow*: Understanding blood flow is crucial for cardiovascular development, identifying cardiovascular diseases, and discovering drug implementations and blood thinners for those at risk.
- *Hurricanes*: Understanding how large-scale flows work are required to know the strength of a hurricane and where it is heading. This is used to help issue warnings for those in danger of being affected by hurricanes!
- *Manufacturing aircraft*: In aviation, we use the fundamental laws of fluid mechanics to help maintain balance and reduce drag in commercial aircraft, which is extremely important when designing their wings or other balance systems!

At the heart of these applications are the Navier-Stokes equations, found everywhere in fluid mechanics. Through derivation, we will see why the Navier-Stokes equations are a universal model for studying fluids.

§2 Review of Vector Fields

Before diving into the Navier-Stokes equations, we proceed with some (hopefully) familiar concepts from vector calculus.

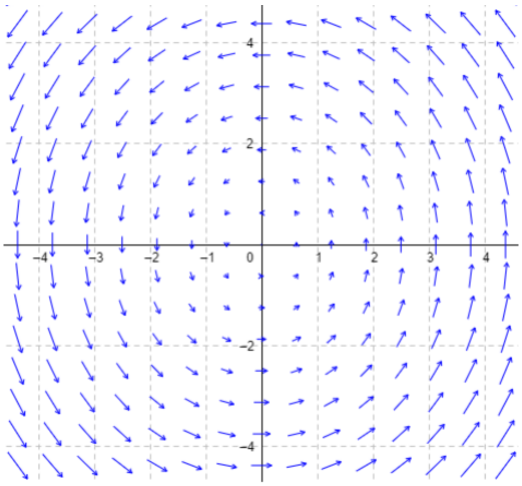
Definition 2.1: Vector Fields

A vector field $\overrightarrow{F}$ takes a point in space and assigns to it a vector. As an example of a vector field in 3 dimensions,

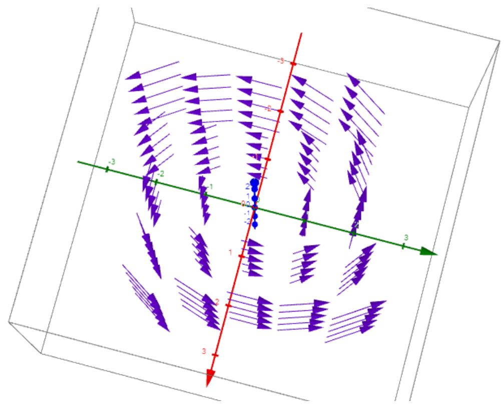
$$(x, y, z) \to \overrightarrow{F} \to P(x, y, z)\hat{i} + Q(x, y, z)\hat{j} + R(x, y, z)\hat{k}$$

where $P, Q,$ and R are functions of the point of interest.

In fluid mechanics, vector fields often represent the velocity of a fluid at a given point (and sometimes, time!), as we will see with the Navier-Stokes equations. For example, consider the vector field $\overrightarrow{F} = (-2y)\hat{i} + (3x)\hat{j}$.



(a) $\overrightarrow{F}$ in 2 dimensions



(b) $\overrightarrow{F}$ in 3 dimensions

Figure 1: Visual implementation of $\overrightarrow{F}$ via geogebra.org

Observe that the vector field moves counterclockwise. In fluid mechanics, we can think of $\overrightarrow{F}$ as the result of stirring a glass of water counterclockwise! Figure 1(a) would describe this flow if we look from a top-down perspective. Notice that the length of a vector corresponds to its *strength*, or how fast a fluid is moving at any point.

Definition 2.2: Flux over a surface

The **flux of a vector field over a surface** is measured by how much of the vector field passes through the surface. The **maximum flux over a surface** is obtained when the vector field is normal, or perpendicular, to the surface at an arbitrary point.

In fluid mechanics, we measure transport flux: the measure how much fluid passes through a surface. The flux depends on not only the orientation of our vector field but our surface as well. For example, we will consider a vector field that is strictly moving to the right, into a cube. Suppose the cube is lying flat (not rotated in the z-direction), as seen in Figure 2. Because the vector field strikes perpendicular to the left surface/face, we obtain the maximum flux! On the contrary, the top surface has no flux because the surface and vector field are parallel!

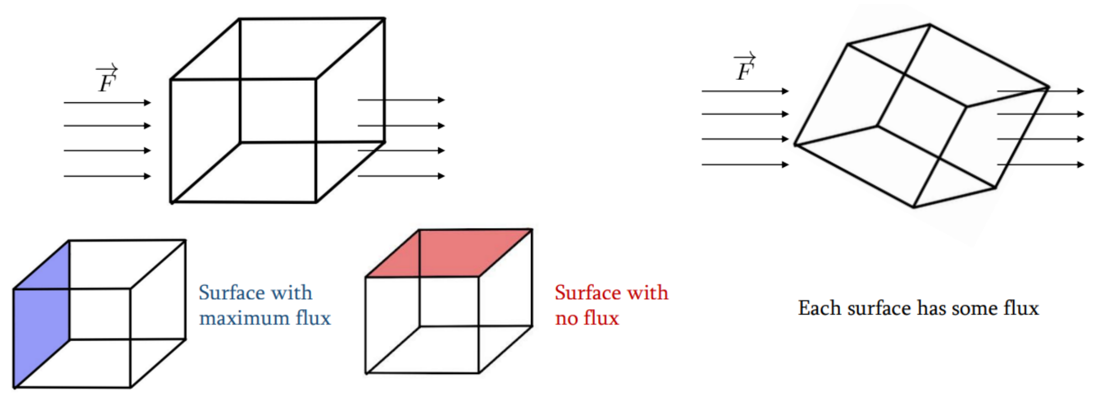


Figure 2: Transport flux depends on the orientation of our surface!

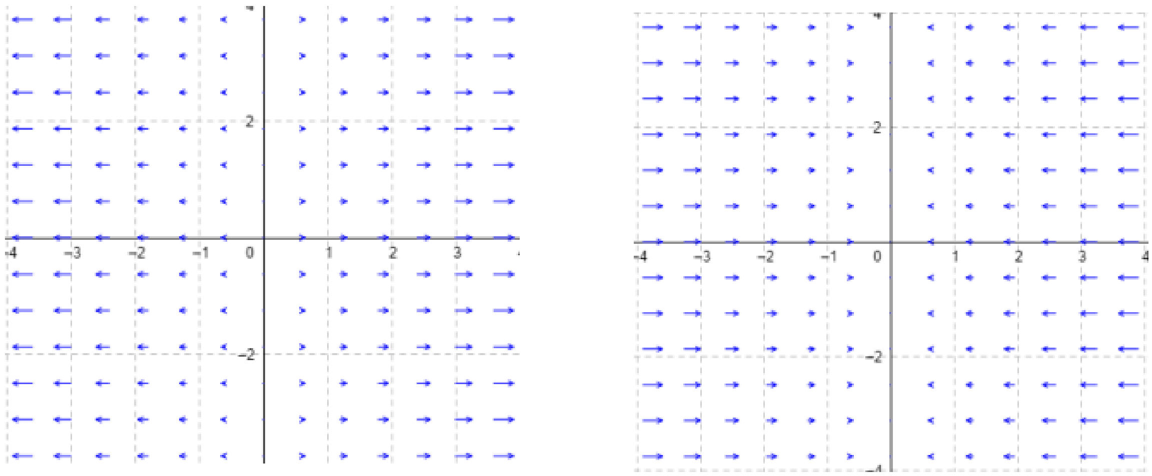
Now, let us consider a rotated cube with the same vector field $\overrightarrow{F}$. Here, each surface receives some flux because the vector field strikes each surface at an angle. We will see the important of flux in understanding how mass of a fluid enters and leaves a three-dimensional space.

Definition 2.3: Divergence of a Vector Field

The **divergence of a vector field** tells us how the strength of the vector field is changing in small neighborhoods surrounding our point. Therefore, we need only look at the sums of our partial derivatives with respect to the corresponding component in our field, yielding a scalar output. So, the divergence is computed by

$$\operatorname{div}\overrightarrow{F} = \nabla \cdot \overrightarrow{F} = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z}$$

In fluid mechanics, divergence models the change in density of a fluid at a given point. Taking a look at Figure 3 (next page), field (a) strengthens away from the origin, yielding a positive divergence. On the other hand, field (b) strengthens towards the origin, yielding a negative divergence.



(a) $\text{div}\overrightarrow{F} > 0$ (b) $\text{div}\overrightarrow{F} < 0$

Figure 3: Divergence comparison

§3 The Navier-Stokes Equations

When studying physics, we recall some of the most conservation theorems: mass, momentum, energy. This is the foundation of the Navier-Stokes equations: the above conservation principles applied to fluids! Now, we consider an overview of the Navier-Stokes equations for an **incompressible** fluid acting over a 3-dimensional domain.

Definition 3.1: The Navier-Stokes Equations

Let $\overrightarrow{V} = \langle u, v, w \rangle$ be the vector field of a fluid acting over a surface. Then, the Navier-Stokes equations for an incompressible fluid are given by:

$$\nabla \cdot \overrightarrow{V} = 0 \qquad \text{Mass Equation}$$

$$\rho \frac{D\overrightarrow{V}}{Dt} = -\nabla p + \mu \nabla^2 \overrightarrow{V} + \rho \overrightarrow{g} \qquad \text{Momentum Equation}$$

The solution $\overrightarrow{V}(x, y, z, t)$ at an arbitrary point in our fluid gives us its velocity components at a specific time. Note that conservation of momentum yields a set of equations, one for each component (x, y, and z).

Keep in mind that the Navier-Stokes Equations can be implemented for any vector field and domain of choice, but we will use a cube and a horizontal vector field to make it simple. Also, while there is an equation for energy, the derivation is very tedious and involved so we are going to omit that part of the derivation.

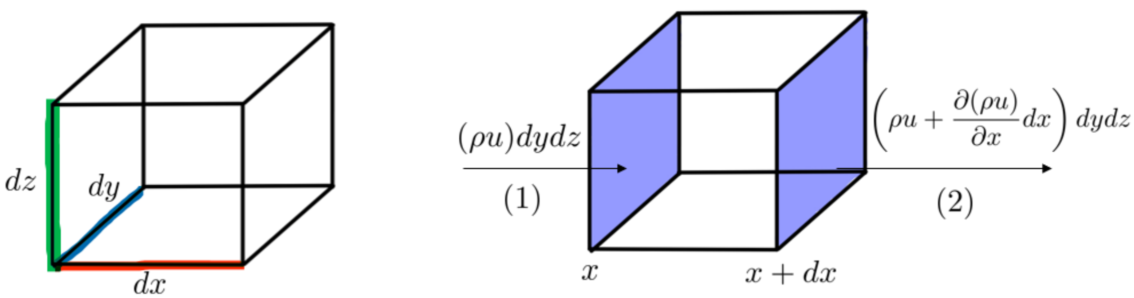
§4 Conservation of Mass Equation

We will look at two definitions of the conservation statement to derive the equation.

Definition 4.1: Conservation of Mass

The following statements are equivalent:
(1) Mass within a domain is conserved.
(2) The mass flux entering each surface is equivalent to the mass flux exiting each surface.

Here we consider an infinitesimally small cube in our fluid, as seen in Figure 4. The length, width, and height are given by dx, dy, dz, respectively (4(a)). We will only see the flux for the x-component, as the flux in the y and z components follow the exact same process. First, we find a way to express the change in mass using statement (1).



(a) Length of each component

(b) Mass Flow Rate for the x-component

Figure 4: Free-body interpretation of Mass Flow Rate

Recall that the mass of a solid is measured by $M = \rho V$, where ρ is the solid's density and V is its volume. Using the formulaic definition of mass and through figure 4(a),

$$M = \rho(dxdydz) = \rho dV \tag{1}$$

Here, dV is constant. Taking the derivative of (1) with respect to time gives

$$\partial M = \frac{\partial \rho}{\partial t} dV \tag{2}$$

Now, we apply the second definition to find an equivalent expression for ∂M. Notice by 4(b) that we consider the left surface at position x, and the right surface at position $x + dx$. The rate in which mass enters (or exits) a surface is defined as the **mass flow rate**, given by

$$\underbrace{\rho}_{\text{density}} \cdot \underbrace{v}_{\text{velocity of fluid}} \cdot \underbrace{A}_{\text{area of surface}} \tag{3}$$

All we need to do is apply this to all 6 surfaces of our cube! We will only show the change in flow in the x-direction as an example. Remember that we use $\overrightarrow{u}$ to denote the x-component of the velocity. The mass flow rate at locations (1) x and (2) $x + dx$ are

$$\partial M_1 = (\rho u) dydz \tag{4}$$
$$\partial M_2 = \left(\rho u + \frac{\partial(\rho u)}{\partial x} dx\right) dydz \tag{5}$$

Notice that for (2), we find that the mass flow rate is computed by the sum of the mass flow rate at position x and the change in mass flow rate from x to $x + dx$.

$$\left(\rho u + \underbrace{\frac{\partial(\rho u)}{\partial x} dx}_{\substack{\text{change in mass flow} \\ \text{from } x \text{ to } dx}}\right) = \text{ mass flow rate at } x + dx$$

We treat the flux entering as a positive quantity and exiting as a negative quantity, so we compute (1) $-$ (2) $= 0$.

$$(\rho u) dydz - \left(\rho u + \frac{\partial(\rho u)}{\partial x} dx\right) dydz = 0 \tag{6}$$
$$(\rho u) dydz - (\rho u) dydz - \frac{\partial(\rho u)}{\partial x} dxdydz = 0 \tag{7}$$
$$\partial M_x = -\rho \frac{\partial u}{\partial x} dV = 0 \tag{8}$$

Note that (8) is the resultant flux for our x-component. What we will see in equation (9) is the sum of fluxes in all components.

$$-\rho \frac{\partial u}{\partial x} dV - \rho \frac{\partial v}{\partial y} dV - \rho \frac{\partial w}{\partial z} dV = -\rho dV \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) = 0 \tag{9}$$

Now we set (9) equal to our mass equation, given in (2).

$$-\rho dV \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) = \frac{\partial \rho}{\partial t} dV \tag{10}$$
$$\underbrace{\frac{\partial \rho}{\partial t} dV}_{=0} + \rho dV \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) \tag{11}$$
$$\nabla \cdot \overrightarrow{V} = 0 \tag{12}$$

In (11), we used our assumption that our fluid is incompressible, so it has no change in density. We can interpret (12) as the absence of change in strength of our vector field, so our fluid is moving at a constant velocity and therefore mass is entering our cube at the same rate it is leaving. Therefore, it has zero density change as we see with the divergence operator.

§5 Conservation of Momentum Equation

Deriving the momentum equations are a bit more involved as we have to look at multiple forces, but again we are only going to make the simplifying assumption that flow is strictly to the right. Also, we will observe that the computation is practically identical to the mass equation!

Definition 5.1: Conservation of Momentum

The following statements are equivalent:
(1) Momentum in our fluid is conserved.
(2) The total change in velocity is equal to the sum of change in pressure, viscosity, and external forces acting upon our fluid.

We start with Newton's Second Law: the sum of forces is directly proportional to mass times acceleration. We will write mass as the product of density and volume, as we did in §4.

$$\sum \overrightarrow{F} = m\overrightarrow{a} = \rho dV \overrightarrow{a} = \rho dV \frac{D\overrightarrow{V}}{Dt} \tag{13}$$

Now, all we need to do is identify the forces acting upon our fluid. We split them into internal and external forces. More precisely,

$$\sum \overrightarrow{F} = \underbrace{-\nabla p}_{\text{change in pressure}} + \underbrace{\mu\nabla^2\overrightarrow{V}}_{\text{viscosity}} + \underbrace{\rho\overrightarrow{g}}_{\text{external forces}} \tag{14}$$

Generally, the only external force observed in our fluid is due to gravity. We will provide more intuition behind pressure and viscosity, and briefly go over gravity. Before looking into our forces, we first provide a more concrete definition of $\frac{D\overrightarrow{V}}{Dt}$, because it is actually not a standard derivative that you find in calculus.

Definition 5.2: Total Derivatives

The term $\frac{D\overrightarrow{V}}{Dt}$, is referred to as a total or "material" derivative, comprised as the sum of two acceleration components in this case: local and convective.

We would often write this derivative as $\frac{d\overrightarrow{V}}{dt}$ because that is what we were taught in calculus. However, total derivatives are used when a particle, or material substance, is subject to a vector field (or flow in a fluid in this case), and is computed as the change in velocity along its pathline. This change in velocity is split into local and convective acceleration, which we will now define.

Definition 5.3: Local and Convective Acceleration

Local acceleration: The acceleration vector experienced by whatever fluid particle is residing at that location and time of interest.
Convective acceleration: The acceleration a fluid particle experiences when it is transported from one location to another.

Picture water flowing down a river. If you focus at one point in the river, you can imagine how fast each particle moves as it passes through that fixed point, which is what local acceleration measures! If you look at a fixed point (x, y, z) in the river and for whatever time t, the acceleration of the particle that is passing through that point yields the local acceleration. Now, picture a rock moving down a river and you track its change in velocity over a set distance. That is what convective acceleration is! Combining definitions 5.2 and 5.3, we get the below relation.

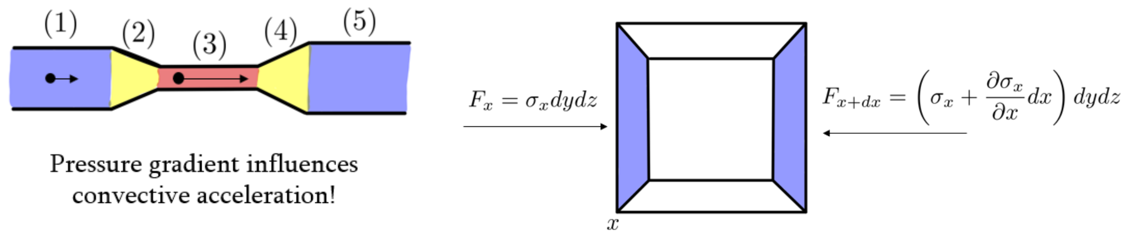
$$\frac{D\overrightarrow{V}}{Dt} = \underbrace{\frac{\partial \overrightarrow{V}}{\partial t}}_{\substack{\text{local}\\ \text{acceleration}}} + \underbrace{(\overrightarrow{V} \cdot \nabla)\overrightarrow{V}}_{\substack{\text{convective}\\ \text{acceleration}}} \tag{15}$$

We think of the total derivative as the sum of the instantaneous change in velocity (local acceleration) and the acceleration components (convective acceleration) at a time t. Now, we take a look at pressure, the first of three forces acting upon our fluid.

Definition 5.4: Pressure acting in a fluid

Pressure is a force that results from applying stress to a fluid. The force is proportional to the product of applied pressure, denoted by σ, and area of impact. Or, $F = \sigma A$.

Taking a look at figure 5(a) in the next page, we can see how pressure influences the convective acceleration of a particle. As the particle moves from (1) to (2), the fluid gets more dense and thus has more pressure, causing it to accelerate. The same goes with (2) to (3), where the particle achieves maximum speed at (3). As the fluid gets less dense from (3) - (5), the particle decelerates from a decrease in pressure.



(a) Pressure gradient (b) Vector intuition of pressure

Figure 5: Pressure acting in a fluid

In 5(b), we consider the exact same setup with a cube. Again, flow is strictly to the right, the area of the left and right surfaces are $A = dydz$, and the applied pressure in the x-direction is σ_x. The calculation for F_x is quite intuitive. As for F_{x+dx}, the same logic is applied when we look at the mass flow rate. We finally compute the net pressure acting upon our fluid.

$$F_{\sigma_x} = F_x - F_{x+dx} = -\frac{\partial \sigma_x}{\partial x}dV \tag{16}$$

Definition 5.5: Viscosity acting in a fluid

Viscosity is the resistance to flow, or the fluid friction, and is generated by shear (sliding) stresses, denoted by τ. Shear forces are proportional to the product of shear stresses applied and the area of the surface. Or, $F = \tau A$.

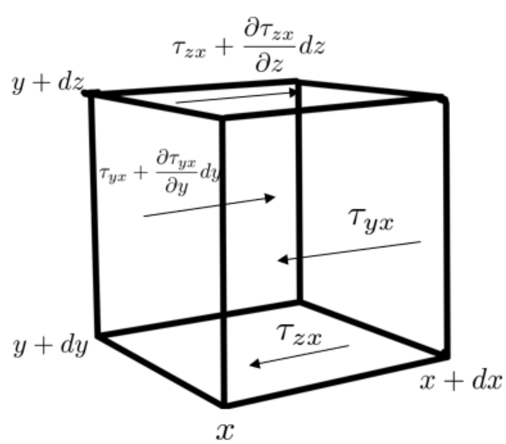


Figure 6: Shear forces acting in the x-direction

If we think of pouring a glass of water compared to a glass of honey, it will take a lot longer for all of the honey to exit the cup. This is because honey has a larger viscosity, or stronger friction with the glass! As the honey exits the glass, there is a forward force as it exits the glass and a strong sliding force in the other direction, causing it to slow down significantly. This is the idea present in Figure 6. Because flow is strictly to the right, we do not observe any friction upon the left and right surfaces. The top and bottom, front and back surfaces experience sliding forces, with the top and back moving in the direction of flow and thus being positive. We first sum every shear stress.

$$\sum \tau = \left(\tau_{yx} + \frac{\partial \tau_{yx}}{\partial y} dy\right) - \tau_{yx} + \left(\tau_{zx} + \frac{\partial \tau_{zx}}{\partial z} dz\right) - \tau_{zx} = \frac{\partial \tau_{yx}}{\partial y} dy + \frac{\partial \tau_{zx}}{\partial z} dz \tag{17}$$

To find the shear force, we multiply each force by their respective surface area.

$$\sum F_\tau = F_{\tau_{yx}} + F_{\tau_{zx}} = \left(\frac{\partial \tau_{yx}}{\partial y} dy\right) dxdz + \left(\frac{\partial \tau_{zx}}{\partial z} dz\right) dxdy = \frac{\partial \tau_{yx}}{\partial y} dV + \frac{\partial \tau_{zx}}{\partial z} dV \tag{18}$$

Last but not least, the weight acting in the x-direction is $F_{gx} = \rho g_x dV$, once again substituting mass for density and volume. Now, we can sum up our forces and set it equal to our total derivative.

$$\sum F = \underbrace{-\frac{\partial \sigma_x}{\partial x} dV}_{\text{pressure}} + \underbrace{\frac{\partial \tau_{yx}}{\partial y} dV + \frac{\partial \tau_{zx}}{\partial z} dV}_{\text{viscosity}} + \underbrace{\rho g_x dV}_{\text{gravity}} = \rho dV \frac{D\overrightarrow{V}}{Dt} \tag{19}$$

The dV terms will cancel. There are ways to relate the pressure and shear forces to the velocity of the fluid, called the constitutive equations for Newtonian flow. Substituting in these relations give us our final momentum equation.

$$\boxed{\begin{aligned} \sigma_x &= -p + 2\mu \frac{\partial u}{\partial x} \\ \tau_{yx} &= \mu \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \\ \tau_{zx} &= \mu \left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) \end{aligned}} \longrightarrow \boxed{\rho \frac{D\overrightarrow{V}}{Dt} = -\nabla p + \mu \nabla^2 \overrightarrow{V} + \rho \overrightarrow{g}}$$

A brief aside on the energy equation. While it will not be derived in this paper, a glimpse into the derivation involves fundamental ideas in thermodynamics: the conservation of energy equation, constituting energy as the work done to the fluid. The energy equation is ultimately influenced by convection, work done by pressure, diffusion, and dissipation.

§6 The Million Dollar Problem

Existence and Uniqueness of Solutions to Navier-Stokes: Millennium Prize Problem

"The challenge is to make substantial progress toward a mathematical theory which will unlock the secrets hidden in the Navier-Stokes equations. A fundamental problem in analysis is to decide whether smooth, physically reasonable solutions exist for the Navier–Stokes equations" (Clay Mathematics Institute, 2000)

The Navier-Stokes Equations are commonly referenced as one of seven million-dollar problems posted by the Clay Mathematics Institute in 2000 (only one has been solved!); the first person successfully able to publish a sufficient, closed proof to the above question will earn one million dollars for their work. The Navier-Stokes equations have been around since 1850, yet remains one of the greatest unsolved puzzles to this day. The Navier-Stokes equations work undisputably because it involves the fundamental laws of physics (Newton's 2nd Law, Conservation of Mass/Momentum/Energy). However, finding a solution that works globally and in every situation has not been found. Why? Like a lot of PDE solutions, we want the solution to (1) exist, (2) be unique, and (3) be smooth. However, we do not even know if a solution exists to every initial condition, nor do we know if it is unique, or if it is smooth! For instance, consider criteria (3). Fluids are prone to turbulence, leading to unexpected changes in velocity. So, even the smallest of changes in our initial condition may lead to turbulence and thus a large change in velocity, which fails to satisfy the smoothness of a solution. Therefore, predicting a solution is often the hardest challenge with Navier-Stokes.

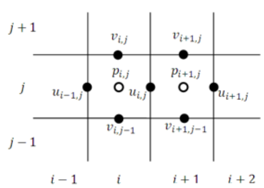
List of Unsolved Millennium Prize Problems

- P versus NP.
- Hodge conjecture.
- Riemann hypothesis.
- Yang–Mills existence and mass gap.
- Navier–Stokes existence and smoothness.
- Birch and Swinnerton-Dyer Conjecture.
- ~~Poincaré conjecture.~~

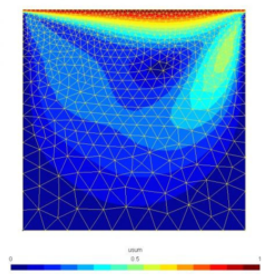
Figure 7: The 7 Millennium Prize Problems proposed by the Clay Mathematics Institute

§7 Approaches to Solving Navier-Stokes

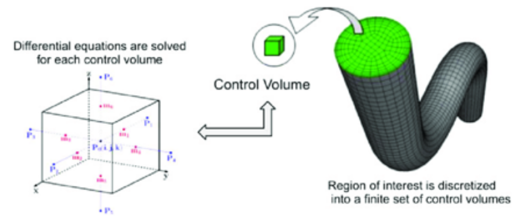
There are ways to solve Navier-Stokes numerically, but it is a very tedious and involved process. Most numerical solutions involve *discretizing* the domain. The main 3 ways are finite differences, finite elements, and finite volume. Each method chops up the fluid into different types of subdomains and solves the PDE within each domain iteratively. Finite differences is often solved using an equally spaced grid (in two dimensions), finite elements can use different shapes as subdomains (triangles, circles, depending on what best fits the domain), and finite volume often partitions the domain into cubes (a three-dimensional grid) of varying size. Figure 8 shows what each method could look like.



(a) Finite differences



(b) Finite elements



(c) Finite volume

Figure 8: Common Algorithms for Discretizing Domains

On the next page, you will find a finite elements implementation of Navier-Stokes in 2 dimensions curated by *Open Source Engineering*, with graphs showing the x and y velocity components, total velocity, and pressure gradient in a rectangular region. This will conclude the paper.

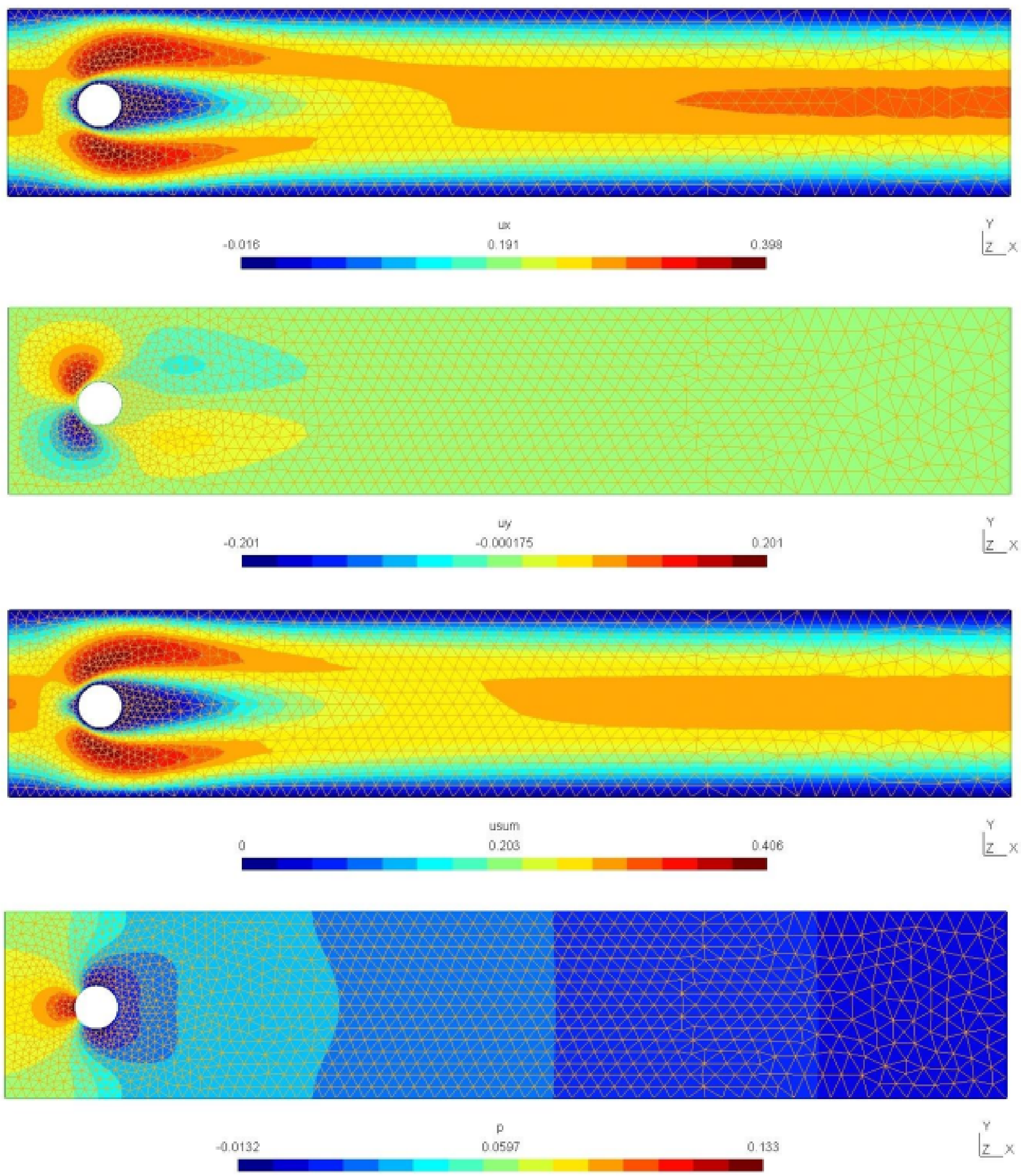


Figure 9: MATLAB Implementation of Navier-Stokes in 2D via Open Source Engineering

Reflection

Overall, this project was the most engaging and enjoyable one I have done in a long time; exploring the boundless realm of PDEs and how unique each of them are made it difficult to pick just one! Ultimately, I settled with the Navier-Stokes equation and did not regret my decision, for it captures the essence of physics and is such a versatile tool for many applications in the real world. Understanding the mathematical and physical intuition behind the derivation was quite the undertaking but I hope it paid off in the presentation and especially in this paper! Although I enjoyed my time working on the project, there are some improvements/additions that I think would have further developed my project. Some of them include, but are not limited to

- A more cohesive presentation. My peers commented that I should have spent more time with the equation and derivation itself rather than the review, considering with the limited time I had. Others remarked that a numerical solution or a conclusion in my presentation to tie in with the numerical approximations section would be more sensible. These are all valid criticisms and I completely agree with them. I tried my best to make these improvements in the paper, and hope to take said criticisms into consideration in future projects.

- If time allowed, I would have liked to gain a thorough understanding of the energy derivation. However, the explanation for energy cannot be easily explained with simple vector addition and requires more knowledge of other useful equations in and laws in thermodynamics, which I do not have a solid background on.

- My own implementation in MATLAB. Unfortunately, a lot of energy was put into the derivation that I ran out of time to develop my own Navier-Stokes simulation. Though I am not sure what initial conditions I would enforce, I would most likely first try using finite differences as that what was discussed in class.

References

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- Million Dollar Problem: https://medium.com/@ases2409/navier-stokes-equations-the-million-dollar-problem-78c01ec05d75
- MATLAB Implementation: http://www.openeering.com/node/18
- Vector Fields: https://www.geogebra.org/m/QPE4PaDZ