Navier-Stokes Equations

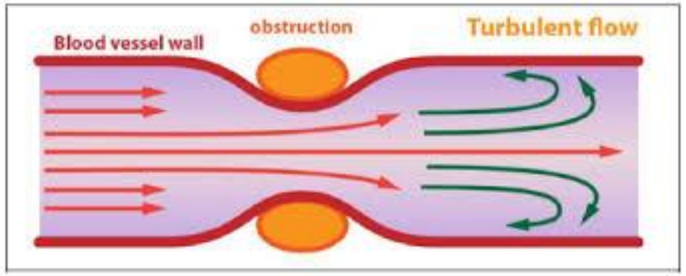
Ryan Gomberg

Contents

Why do we study Fluid Mechanics?

At its core, fluid mechanics studies fluids in motion. Despite the simple definition, modeling fluids in the real world is complicated and arguably one of the hardest fields in physics to study due to the difficulty in predicting patterns.

Nonetheless, the applications in fluid mechanics are invaluable.



The crux of these applications are the Navier-Stokes equations! Through derivation, we will see how the Navier-Stokes equations is a universal model of fluid mechanics.

Review of Vector Fields

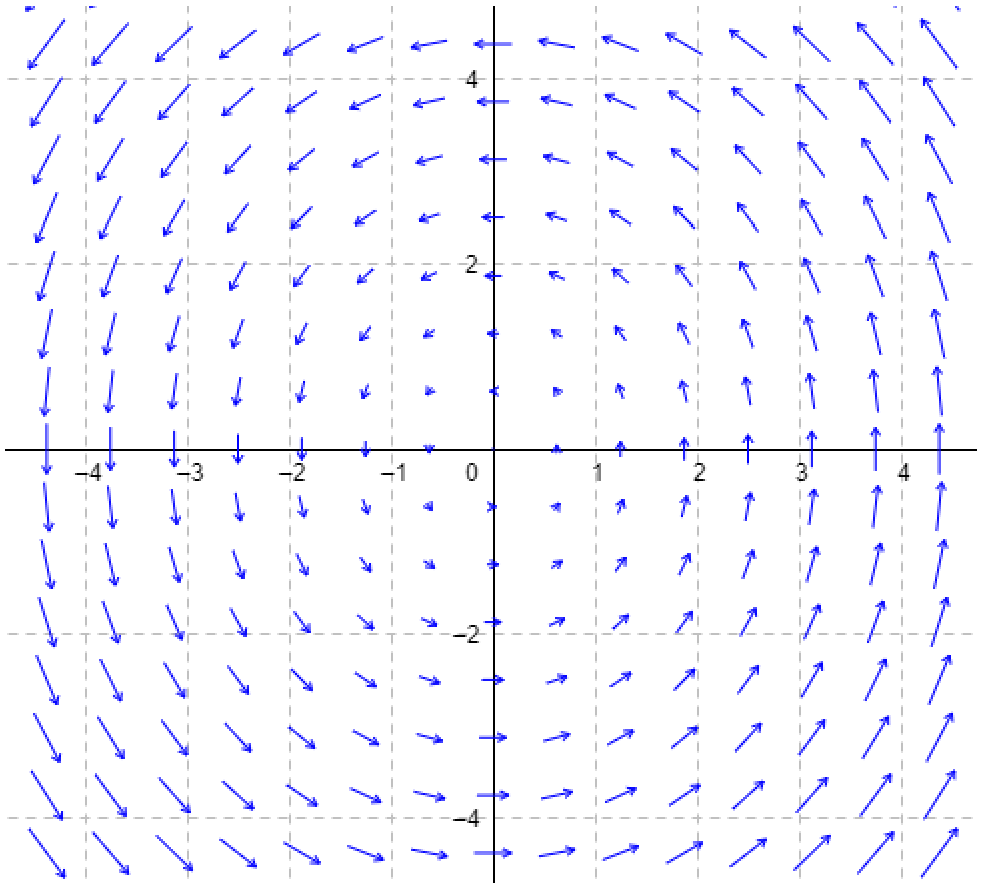
A vector field $\overrightarrow{F}$ takes a point in space and assigns it to a vector. For example, in 3 dimensions, we observe

$$\boxed{(x, y, z) \rightarrow \overrightarrow{F} \rightarrow P(x, y, z)\hat{i} + Q(x, y, z)\hat{j} + R(x, y, z)\hat{k}}$$

In fluid mechanics, vector fields often represent the velocity of a fluid at a given point, as we will see with the Navier-Stokes equations.

As an example, to the right is a two-dimensional vector field. Notice that the length of the vector at any point corresponds to its "strength."

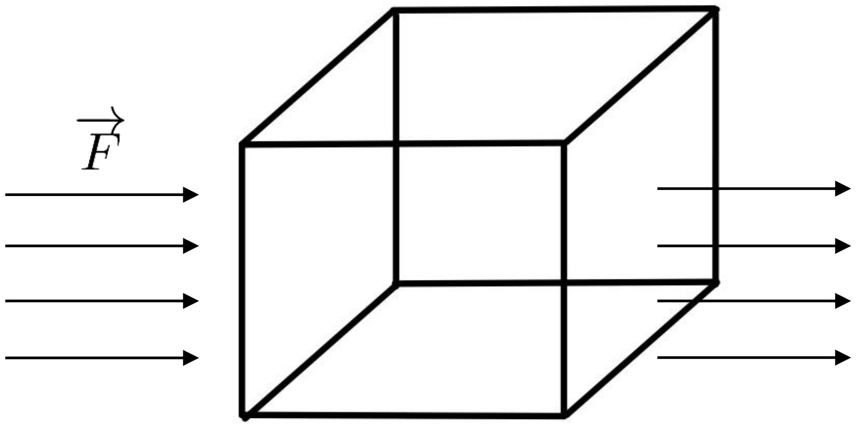
$$\overrightarrow{F} = (-2y)\hat{i} + (3x)\hat{j}$$

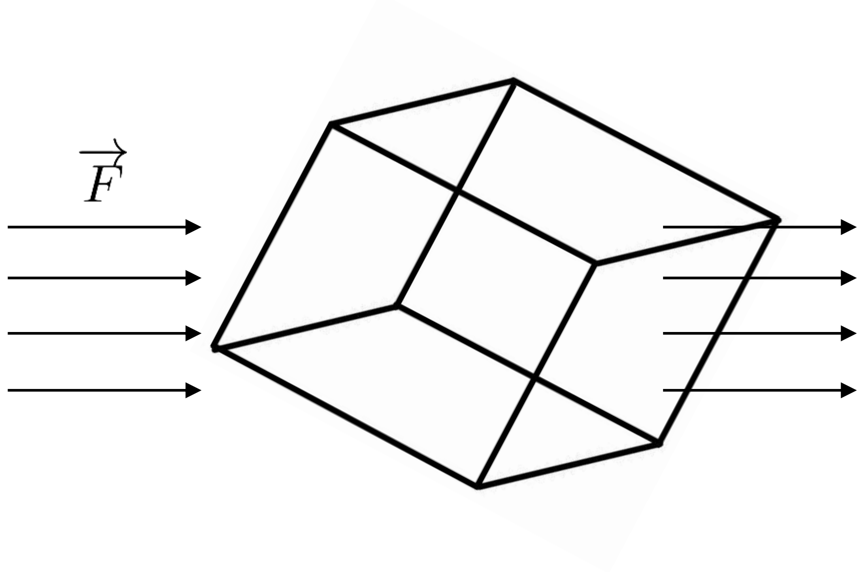


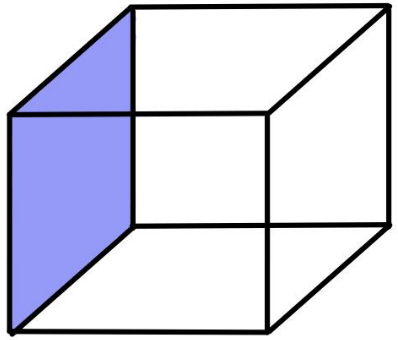
Obtained via geogebra.org

Review of Vector Fields – Flux

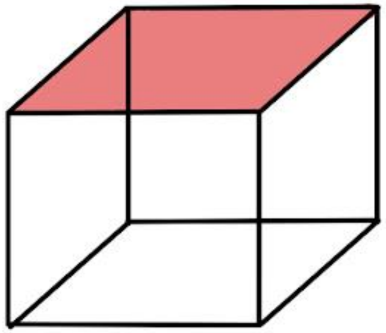
Flux over a surface - We say the flux of a vector field over a surface is measured by how much of the vector field passes through the surface. In fluid mechanics, transport flux measures how much fluid passes through a surface.







Surface with maximum flux



Surface with no flux

Each surface has some flux

Review of Vector Fields – Divergence

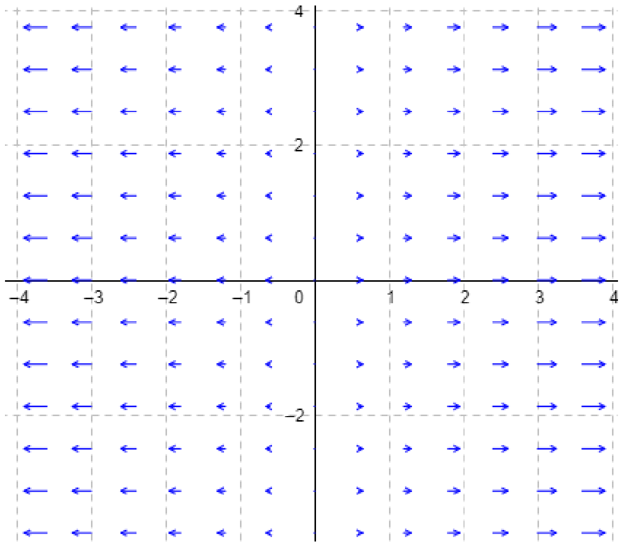
Divergence - Suppose we are given a vector field

$$\overrightarrow{F} = \langle f, g, h \rangle$$

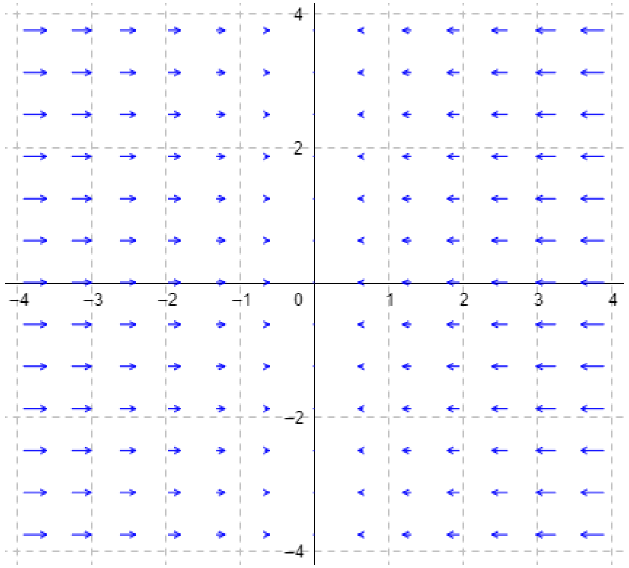
The divergence of a vector field at a given point tells us how the strength of the vector field is changing in small neighborhoods surrounding our point. Therefore, we need only look at the sums of our partial derivatives with respect to the corresponding component in our field, yielding a scalar output. So, the divergence is computed by

$$\operatorname{div} \overrightarrow{F} = \nabla \cdot \overrightarrow{F} = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z}$$

In fluid mechanics, divergence models the change in density of a fluid at a given point. For incompressible fluids, its divergence is zero.



$$\operatorname{div} \overrightarrow{F} > 0$$



$$\operatorname{div} \overrightarrow{F} < 0$$

The Navier-Stokes Equations (for incompressible fluids)

When studying physics, we recall some of the most conservation theorems: **mass, momentum, energy**. This is the foundation of the Navier-Stokes equations: the above conservation principles applied to fluids!

In particular, we will derive the mass and momentum equations for flow over a 3-dimensional domain. Let $\overrightarrow{V} = \langle u, v, w \rangle$ be the vector field of a fluid acting over a surface. Then, the Navier-Stokes equations are given by:

$\nabla \cdot \overrightarrow{V} = 0$	Conservation of Mass Equation (1)
$\rho \frac{D\overrightarrow{V}}{Dt} = -\nabla p + \mu \nabla^2 \overrightarrow{V} + \rho \overrightarrow{g}$	Conservation of Momentum Equations (2 – 4)

The solution $\overrightarrow{V}(x, y, z, t)$ at an arbitrary point in our fluid gives us its velocity components at a specific time.

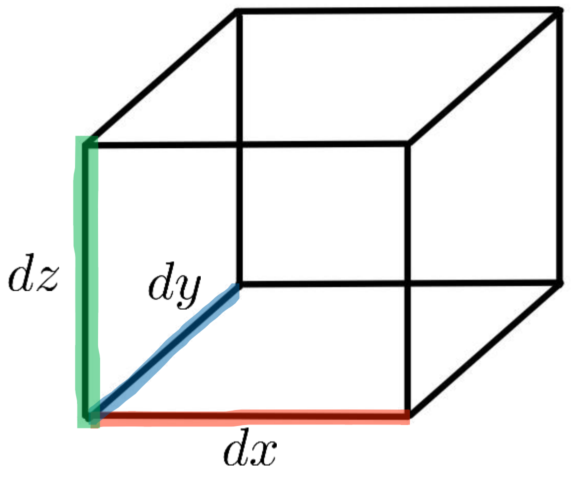
Note that conservation of momentum yields a set of equations, one for each component (x, y, and z).

Conservation of Mass Equation

Statement: The mass in our domain is conserved; or more precisely, the change in mass of our domain is equal to its change in flux across every boundary. For simplicity, we will let our domain be a cube, denoted by C. We can write the conservation statement as such:

$$\partial M_C = 0$$

We use the formulaic definition of mass to find an expression for ∂M_C, as shown below.



$$M_C = \rho dxdydz$$

(1) Volume of our domain = density * volume

$$\partial M_C = \frac{\partial \rho}{\partial t} dxdydz$$

(2) Taking the derivative yields change in mass

$$= \frac{\partial \rho}{\partial t} dV$$

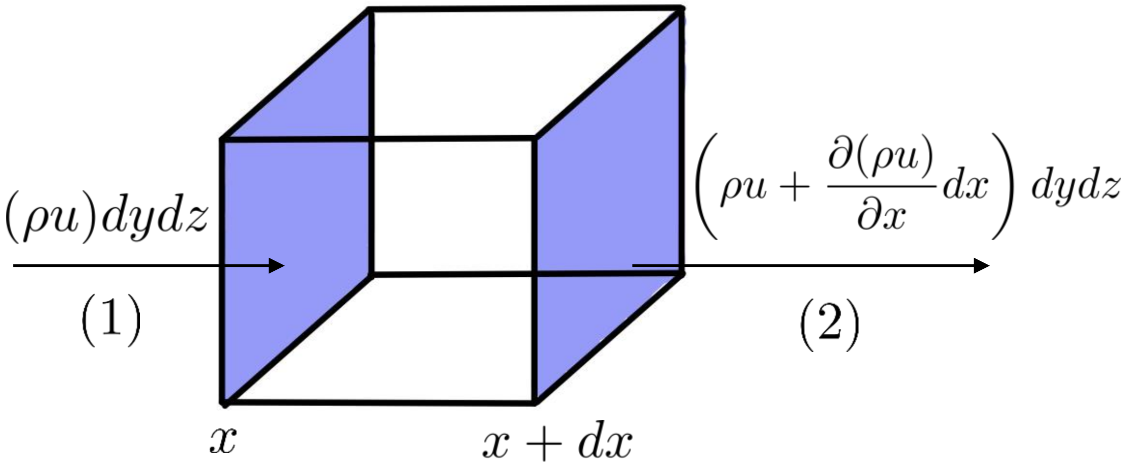
Now, we will use the definition of flux to find an alternative expression for ∂M_C.

Conservation of Mass Equation

Recall the statement: the change in mass is equal to the sum of fluxes entering (or exiting) every boundary. The rate in which mass enters a surface (mass flow rate) is given by

$$\underbrace{\rho}_{\text{density}} \cdot \underbrace{v}_{\text{velocity of fluid}} \cdot \underbrace{A}_{\text{area of surface}}$$

All we need to do is apply this to all 6 surfaces of our cube! We will only show the change in flow in the x-direction as an example. Remember that we use $\overrightarrow{u}$ to denote the x-component of the velocity.



$$\left(\rho u + \underbrace{\frac{\partial \rho u}{\partial x}dx}_{\substack{\text{change in mass flow}\\ \text{from } x \text{ to } dx}}\right) = \text{mass flow rate at } x + dx$$

To maintain conservation, we require (1) and (2) to be equal, or (1) – (2) = 0.

$$(\rho u)dydz - \left(\rho u + \frac{\partial(\rho u)}{\partial x}dx\right)dydz = 0$$

$$\cancel{(\rho u)dydz - (\rho u)dydz} - \frac{\partial(\rho u)}{\partial x}dxdydz = 0$$

$$-\rho\frac{\partial u}{\partial x}dV = 0$$

Conservation of Mass Equation

The derivation for the y and z components are exactly the same. So, the change in flux for our cube is

$$-\rho\frac{\partial u}{\partial x}dV - \rho\frac{\partial v}{\partial y}dV - \rho\frac{\partial w}{\partial z}dV = -\rho dV\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) = 0$$

We set the equations equal to each other

$$-\rho dV\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) = \frac{\partial \rho}{\partial t}dV$$

$\frac{\partial \rho}{\partial t}$ = 0 by incompressibility

$$\cancel{\frac{\partial \rho}{\partial t}}dV + \rho dV\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) = 0$$

$$\nabla \cdot \overrightarrow{V} = 0$$

Conservation of Momentum Equation

We proceed by recalling the formulaic expression for Newton's Second Law

$$\sum \overrightarrow{F} = m\overrightarrow{a}$$

By the formulaic definition of mass, this becomes

$$\sum \overrightarrow{F} = \rho dV \overrightarrow{a}$$

Acceleration is the time derivative of velocity, so the sum of forces acting on an object is equal to its the change in momentum. Because $\overrightarrow{V}$ represents the velocity components of a fluid, we can write

$$\sum \overrightarrow{F} = \rho dV \frac{D\overrightarrow{V}}{Dt}$$

Now, all we need to do is identify the forces acting upon our domain. We split them into internal and external forces. More precisely,

$$\sum \overrightarrow{F} = \underbrace{-\nabla p}_{\substack{\text{change in} \\ \text{pressure}}} + \underbrace{\mu \nabla^2 \overrightarrow{V}}_{\text{viscosity}} + \underbrace{\rho \overrightarrow{g}}_{\substack{\text{external} \\ \text{forces}}}$$

We are going to derive these in a scenario in which flow is only occurring in the x-direction.

Local and Convective Acceleration

The term $\dfrac{D\overrightarrow{V}}{\partial t}$ is referred to as a total or "material" derivative, comprised as the sum of two acceleration components: local and convective

Local acceleration: The acceleration vector experienced by whatever fluid particle is residing at that location and time of interest.

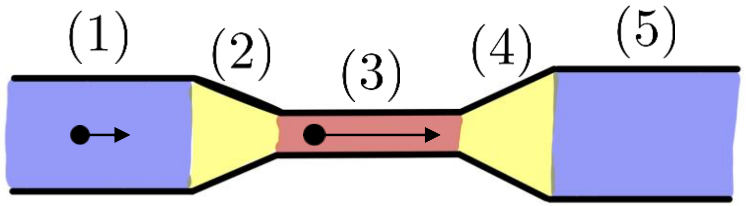
Convective acceleration: The acceleration a fluid particle experiences when it is transported from one location to another.

We have the following relation

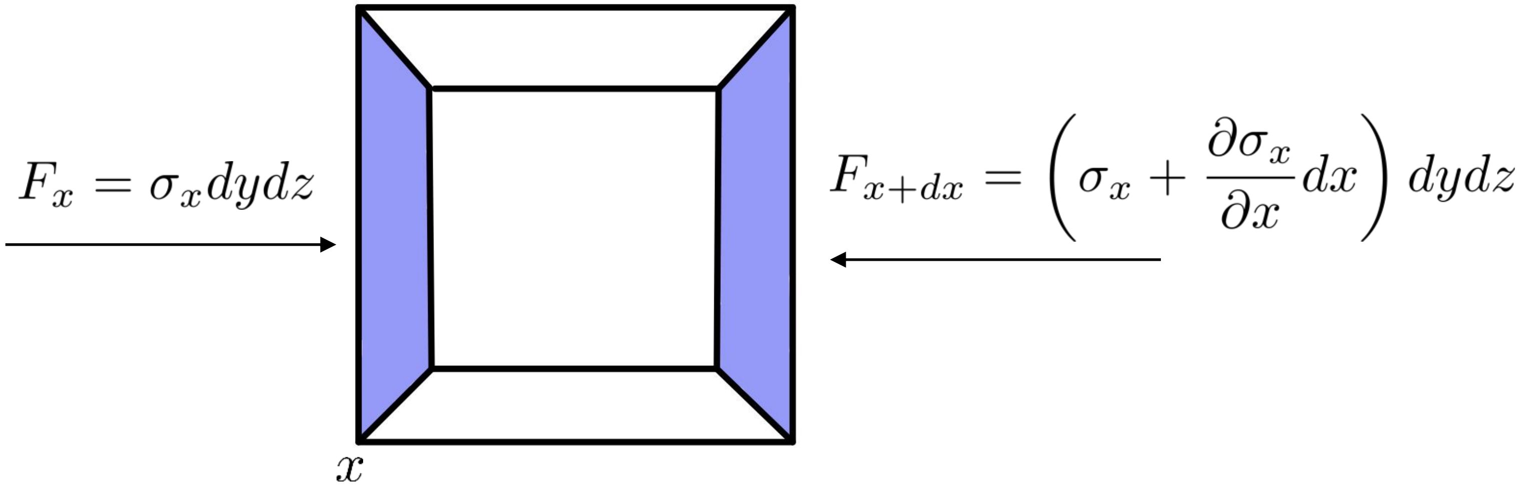
$$\frac{D\overrightarrow{V}}{Dt} = \underbrace{\frac{\partial \overrightarrow{V}}{\partial t}}_{\substack{\text{local}\\ \text{acceleration}}} + \underbrace{(\overrightarrow{V}\cdot\nabla)\overrightarrow{V}}_{\substack{\text{convective}\\ \text{acceleration}}}$$

Conservation of Momentum – Pressure

Pressure is a force that results from applying stress to a fluid. The force is proportional to the product of applied pressure, denoted by σ_x, and area of impact. Or, $F = \sigma A$.



Pressure gradient influences convective acceleration!

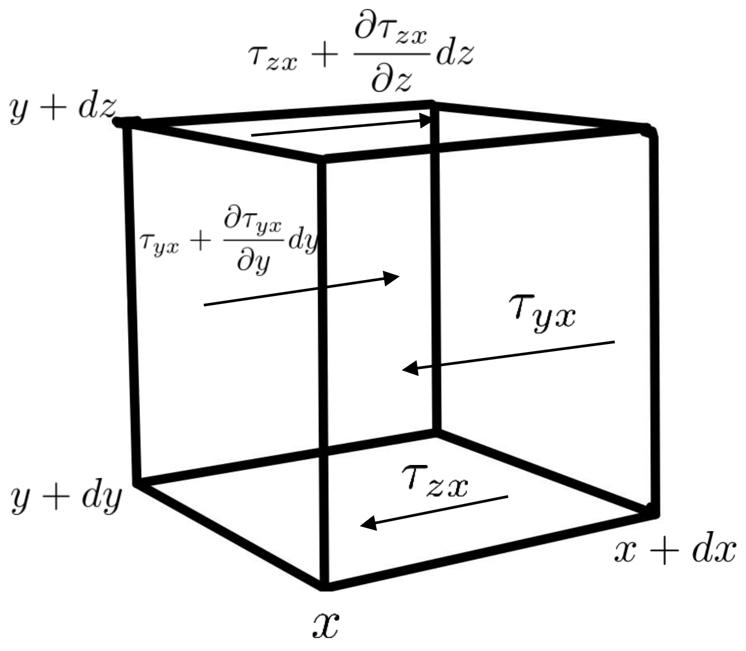


Computing the net force is practically identical to the net flux calculation we did earlier:

$$F_{\sigma_x} = F_x - F_{x+dx} = -\frac{\partial \sigma_x}{\partial x} dV$$

Conservation of Momentum – Viscosity

In fluid mechanics, viscosity is the resistance to flow, or the fluid friction. For example, pouring a glass of water is a lot faster than pouring a glass of honey. This is because honey has a significantly larger viscosity! Viscosity is defined by **shear (sliding) stresses**, denoted by τ. Shear forces are proportional to the product of shear stresses applied and the area of the surface. Or, $F = \tau A$. We first sum every shear stress.



$$\sum \tau = \left(\tau_{yx} + \frac{\partial \tau_{yx}}{\partial y} dy\right) - \tau_{yx} + \left(\tau_{zx} + \frac{\partial \tau_{zx}}{\partial z} dz\right) - \tau_{zx}$$

$$= \frac{\partial \tau_{yx}}{\partial y} dy + \frac{\partial \tau_{zx}}{\partial z} dz$$

To find the shear force, we multiply each force by their respective surface.

$$\sum F_\tau = F_{\tau_{yx}} + F_{\tau_{zx}} = \left(\frac{\partial \tau_{yx}}{\partial y} dy\right) dxdz + \left(\frac{\partial \tau_{zx}}{\partial z} dz\right) dxdy$$

$$= \frac{\partial \tau_{yx}}{\partial y} dV + \frac{\partial \tau_{zx}}{\partial z} dV$$

Conservation of Momentum Equation

Quickly, we assume that the only external force is the weight acting upon the fluid. So, the weight in the x-direction is $F_{gx} = \rho g_x dV$. Now, we can sum up our forces and set it equal to our total derivative.

$$\sum F = \underbrace{-\frac{\partial \sigma_x}{\partial x} d\cancel{V}}_{\text{pressure}} + \underbrace{\frac{\partial \tau_{yx}}{\partial y} d\cancel{V} + \frac{\partial \tau_{zx}}{\partial z} d\cancel{V}}_{\text{viscosity}} + \underbrace{\rho g_x d\cancel{V}}_{\text{gravity}} = \rho d\cancel{V} \frac{D\overrightarrow{V}}{Dt}$$

There are ways to relate the pressure and shear forces to the velocity of the fluid, called the **constitutive equations for Newtonian flow**. Substituting in these relations give us our final momentum equation.

$$\boxed{\begin{aligned} \sigma_x &= -p + 2\mu \frac{\partial u}{\partial x} \\ \tau_{yx} &= \mu \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \\ \tau_{zx} &= \mu \left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \right) \end{aligned}} \longrightarrow \boxed{\rho \frac{D\overrightarrow{V}}{Dt} = -\nabla p + \mu \nabla^2 \overrightarrow{V} + \rho \overrightarrow{g}}$$

The "Million-Dollar Problem"

> "The challenge is to make substantial progress toward a mathematical theory which will unlock the secrets hidden in the Navier-Stokes equations. A fundamental problem in analysis is to decide whether smooth, physically reasonable solutions exist for the Navier–Stokes equations."
>
> Clay Mathematics Institute (2000)

The Navier-Stokes equations have been around since 1850, yet remains one of the greatest unsolved puzzles to this day. The Navier-Stokes equations work undisputably because it involves the fundamental laws of physics (Newton's 2^{nd} Law, Conservation of Mass/Momentum/Energy). However, finding a solution that works globally and in every situation has not been found. Why?

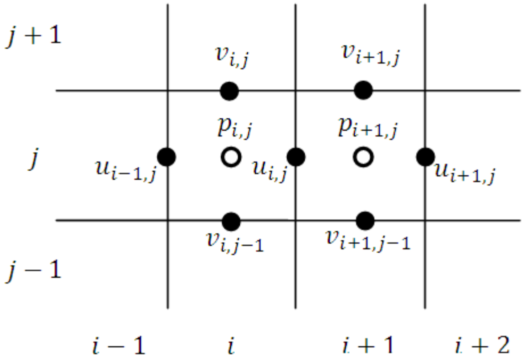
We want the solution to (1) exist, (2) be unique, and (3) be smooth. However, we do not even know if a solution exists to every initial condition!

One difficulty lies in predicting the solution. Fluids are prone to turbulence, leading to unexpected changes in velocity. So, even the tiniest changes in our initial condition may lead to turbulence and thus a large change in velocity, which fails to satisfy the smoothness of a solution.

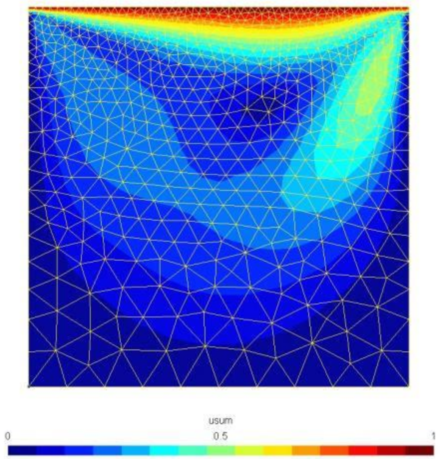
Approaches to Solving Navier-Stokes

There are ways to solve Navier-Stokes numerically, but it is a very tedious and involved process. Most numerical solutions involve "discretizing" the domain. The main 3 ways are

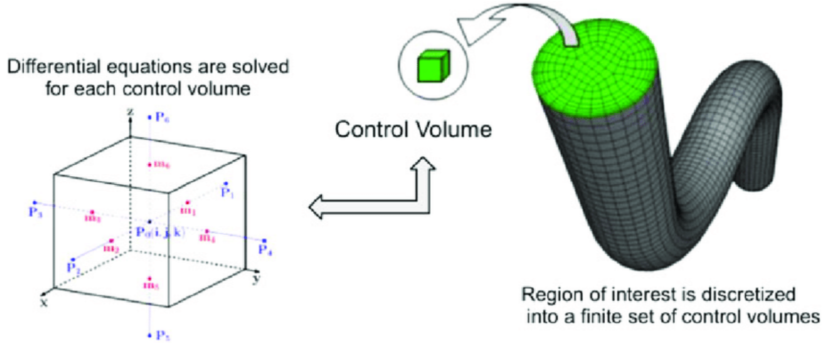
Finite difference



Finite elements



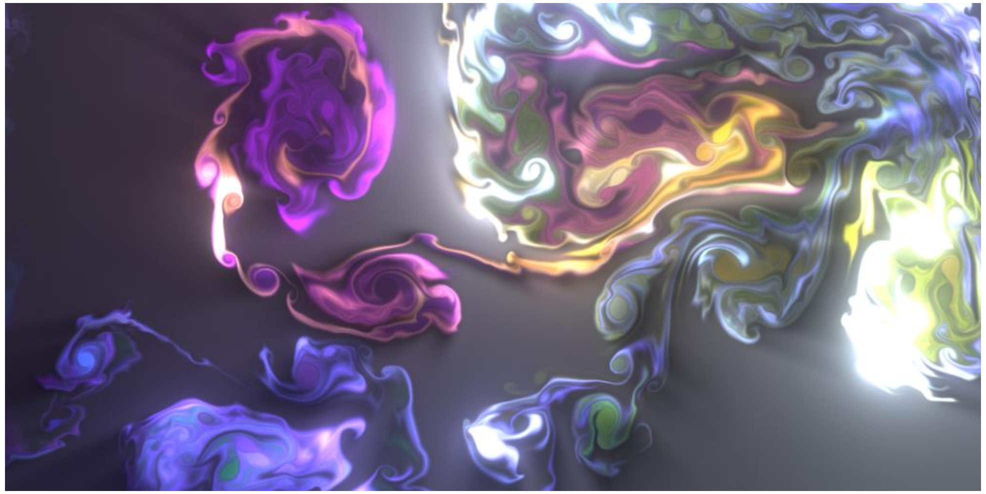
Finite volume
Most preferred
Works well with conservation laws



Other ways involve simplifying assumptions (for example, ignoring time in the problem).

Simulations of Navier-Stokes

The following website was used to generate the below simulation:
https://www.outpan.com/app/44bdd9869c/interactive-fluid-simulation



References

1. Derivation of Mass-Continuity Equation: https://www.youtube.com/watch?v=v9Y_O74_fV0
2. Navier-Stokes Equation: https://www.youtube.com/watch?v=ERBVFcutl3M
3. Pressure & Shear Forces: https://www.tec-science.com/mechanics/gases-and-liquids/derivation-of-the-navier-stokes-equations/
4. Constitutive equations for Newtonian Flow: https://www.youtube.com/watch?v=NjoMoH51UZc&t=12s
5. "Million Dollar Problem:" https://medium.com/@ases2409/navier-stokes-equations-the-million-dollar-problem-78c01ec05d75
6. Simulations: https://www.outpan.com/app/44bdd9869c/interactive-fluid-simulation