1 Properties of Complex Numbers

Definition 1.1: Complex Numbers and Properties

A complex number z is of the form $z = x + iy$, where $x, y \in \mathbb{R}$.

The *modulus* $|z| = \sqrt{x^2 + y^2}$ is the distance from the origin to z and the *conjugate* of z, $\overline{z}$, is $\overline{z} = z - iy$.

Example 1: Show that $|z|^2 = z\overline{z}$.

Simply apply the definitions:

$$|z|^2 = x^2 + y^2, z\overline{z} = (x + iy)(x - iy) = x^2 + y^2 \implies |z|^2 = z\overline{z}.$$

Definition 1.2: Complex Numbers in Polar Form, Euler's Form

The polar form of $z = x + iy$ is the form

$$z = r\cos\theta + ir\sin\theta = r(\cos\theta + i\sin\theta)$$

where $r = |z|$ is the modulus of z and θ is the *argument* of z. The principal argument Arg$z$ spans $-\pi \leq \text{Arg} \leq \pi$.

Euler's form of z is of the form $z = re^{i\theta} = |z|e^{i\theta} = r(\cos\theta + i\sin\theta)$

Definition 1.3: Roots of Complex Numbers

Given a non-zero complex number $z = re^{i\theta}$ and a positive integer n, the n^{th} roots of z are the n distinct complex numbers

$$c_k = \sqrt[n]{r}e^{\frac{(\theta+2k\pi)i}{n}}, k = 0, 1, ..., n - 1$$

where $\sqrt[n]{z} = \sqrt[n]{r}e^{\frac{i\theta}{n}}$ is the *principal n^{th} root.*

2 Holomorphic Functions

Functions of a Complex Variable

Example 2: Compute the function that reflects across the line joining $\alpha = 2 + i$ and $\beta = 4 + 3i$.

Limits and Continuity

Many of the laws and definitions about limits/continuity from real analysis also apply to complex numbers.

Definition 2.3: Continuity

Let $f : D \to \mathbb{C}$ and $z_0 \in D$. We say f is *continuous at* z_0 if

$$\text{For all sequences } (z_n) \subseteq D \text{ with } \lim z_n = z_0 \text{ we have } \lim f(z_n) = f(z_0)$$

f is *continuous* (on D) if it is continuous at all points $z_0 \in D$.

As we will see with the limit definition, we are now thinking of continuity over a neighborhood, or more precisely, a domain containing open disks. An *punctured* open disk has a radius $0 < |z - z_0| < \delta$ for some $\delta > 0$, $z \in \mathbb{C}$.

Definition 2.4: Limits of Complex Functions

Let $f : D \to \mathbb{C}$, where D contains an open punctured neighborhood of z_0. We say that w_0 is the *limit of* f *as* z *approaches* z_0, written $\lim_{z \to z_0} f(z) = w_0$, if

$$\forall \epsilon > 0, \exists \delta > 0 \text{ such that } 0 < |z - z_0| < \delta \implies |f(z) - w_0| < \epsilon$$

Note that, unlike in real analysis, the bars are actually notation for the modulus, as opposed to absolute value, as distance is measured in 2 dimensions. We need to ensure that the distance between $f(z)$ and w_0 between z and z_0 grows infinitesimally small as z and z_0 get closer.

Example 3: Show that $\lim_{z \to 0} \frac{\overline{z}^2}{z} = 0$.

Proof: Fix $\epsilon > 0$ and let $\delta = \epsilon$. Then, we have that

$$\left| \frac{\overline{z}^2}{z} - 0 \right| < \epsilon \text{ given } |z - 0| < \delta \implies \left| \frac{\overline{z}^2}{z} \right| = \frac{|\overline{z}^2|}{|z|} = \frac{|z|^2}{|z|} = |z| < \epsilon = \delta.$$

Example 4: Prove $\lim_{z \to z_0} z^3 = z_0^3$ using the ϵ-δ definition.

Sketch Proof: We have that $|z^3 - z_0^3| < \epsilon$ for $|z - z_0| < \delta$. We are able to use the difference of cubes to rewrite $|z^3 - z_0^3|$ as

$$|z^3 - z_0^3| = |(z - z_0)(z^2 + zz_0 + z_0^2)|$$

Now, we have to find a bound for $(z^2 + zz_0 + z_0^2)$. Let us assume $|z - z_0| < 1$, then

$$|z| = |z - z_0 + z_0| \leq |z - z_0| + |z_0| \text{ (by triangle inequality) } < 1 + |z_0| \text{ by assumption.}$$

This enables us to find a δ directly in terms of z_0. Hence,

$$|z^3 - z_0^3| < |z - z_0|(1 + |z_0|)^2 + |z_0|(1 + |z_0|) + |z_0|^2) < \epsilon$$

Therefore, suppose

$$\delta = \min\left\{1, \frac{\epsilon}{(1 + |z_0|)^2 + |z_0|(1 + |z_0|) + |z_0|^2)}\right\}$$

completing the proof.

Example 5: Suppose $\lim_{z \to z_0} f(z) = w_0$. Prove that $\lim_{z \to z_0} |f(z)| = |w_0|$.

By our assumption, we have that $|f(z) - w_0| < \epsilon$ for $|z - z_0| < \delta$. Now, let $\epsilon_2 > 0$. We want to find a $\delta_2 > 0$ such that

$$|z - z_0| < \delta_2 \Longleftarrow ||f(z)| - |w_0|| < \epsilon_2.$$

The reverse triangle inequality gives us

$$||f(z)| - |w_0|| \leq |f(z) - w_0| < \epsilon = \epsilon_2$$

Because $\epsilon_2 = \epsilon$, we simply choose $\delta_2 = \delta$.

Definition 2.5: Limits Approaching Infinity

A *neighborhood* of ∞ is any set containing an *open disk* at ∞, a subset of the form $\{\infty\} \cup \{z \in \mathbb{C} : |z| > M\}$. Note that we only care about one infinity in complex analysis; negative infinities do not exist. Here, we make the following implications

$$\lim_{z \to z_0} f(z) \text{ means: } \forall M > 0, \exists \delta > 0 \text{ such that } 0 < |z - z_0| < \delta \Longrightarrow |f(z)| > M$$

$$\lim_{z \to \infty} f(z) = w_0 \text{ means: } \forall \epsilon > 0, \exists N > 0 \text{ such that } |z| > N \Longrightarrow |f(z) - w_0| < \epsilon$$

$$\lim_{z \to \infty} f(z) = \infty \text{ means: } \forall M > 0, \exists N > 0 \text{ such that } |z| > N \Longrightarrow |f(z)| > M$$

Example 6: Prove $\lim_{z\to -3i} \frac{z^2}{z+3i} = \infty$.

Proof: Assume $\delta < 1$. We have that

$$0 < |z+3i| < \delta \implies |z| \geq |3i| - |z+3i| \text{ (by triangle ineq) } > 3-\delta \geq 2$$

$$\implies \left|\frac{z^2}{z+3i}\right| > \left|\frac{4}{\delta}\right| \geq M \iff \left|\frac{4}{\delta}\right| \geq M$$

Therefore, choose $\delta = \min\left\{1, \frac{4}{M}\right\}$.

Example 7: Prove $\lim_{z\to\infty} \frac{iz-1}{z-2i} = i$.

Proof: Let $\epsilon > 0$ be fixed. Then, there exists a $N > 0$ such that

$$\left|\frac{iz-1}{z-2i} - i\right| = \left|\frac{-3}{z-2i}\right| = \frac{3}{|z-2i|} < \epsilon.$$

To find such N, we compute

$$|z-2i| = |z| - |2i| = |z| - 2 > N \implies |z| > N \text{ when } N > 2$$

Therefore,

$$\frac{|z|-2}{3} > \frac{1}{\epsilon} \iff |z| > \frac{3}{\epsilon} + 2 = N \text{ for } N > 2$$

Theorem 2.6: Cauchy-Riemann Equations

Let $f(z)$ be decomposed into $f(x+iy)$. If $f(z) = u + iv$ if differentiable at z_0, then its real/imaginary parts, in terms of x, y, satisfy the following relations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

For such z_0, its derivative is computed as $f'(z_0) = u_x + iv_x|_{(x,y)}$.
If the relation holds for all $x, y \in \mathbb{R}$, then $f(z)$ is everywhere differentiable.

Example 8: Using the Cauchy-Riemann Equations, show that $f(z) = z^3 - \frac{2}{z}$ is differentiable everywhere except for when $z = 0$ and find its derivative.

Rewriting $z = x + iy$, we have

$$(x+iy)^3 - \frac{2}{x+iy} = x^3 + 3ix^2y - 3xy^2 - iy^3 - \frac{2(x-iy)}{x^2+y^2} = \left(x^3 - 3xy^2 - \frac{2x}{x^2+y^2}\right) + \left(3x^2y - y^3 - \frac{2y}{x^2+y^2}\right)i$$

Here, u is the first set of expressions inside the parenthesis, and v is the coefficient of i. Now, we show $u_x = v_y$, $u_y = -v_x$.

$$u_x = 3x^2 - 3y^2 - \frac{2y^2 - 2x^2}{(x^2+y^2)^2} \qquad v_y = 3x^2 - 3y^2 + \frac{2x^2 - 2y^2}{(x^2+y^2)^2}$$

$$u_y = -6xy + \frac{4xy}{(x^2+y^2)^2} \qquad -v_x = -6xy + \frac{4xy}{(x^2+y^2)^2}$$

Since both criteria are satisfied, we can conclude $f(z)$ is differentiable on $\mathbb{C}\backslash\{0\}$ and $f'(z) = 3z^2 + \frac{2}{z^2}$, as we expected.

Example 9: Find the set of points in which $f(z) = (|z|^2 + z)^2$ is differentiable. If such points exist, find their derivative.

$$(|x+iy|^2 + (x+iy))^2 = ((x^2+y^2) + (x+iy))^2$$
$$= (x^4 + 2x^3 + x^2 + 2x^2y^2 + 2y^2x + y^4 - y^2) + 2(x^2y + xy + y^3)i$$

Setting u to the first parentheses and v to the second

$$u_x = 4x^3 + 6x^2 + 2x + 4xy^2 + 2y^2 \qquad v_y = 2x^2 + 2x + 6y^2$$

$$u_y = 4x^2y + 4yx + 4y^3 - 2y \qquad -v_x = -4xy - 2y$$

We obtain a system of equations $u_x - v_y = 0$ and $u_y - (-v_x) = 0$

$$\begin{cases} u_x - v_y = 4x^3 + 4x^2 + 4xy^2 - 4y^2 = 0 \iff x^2(x+1) = y^2(1-x) \\ u_y - (-v_x) = 4x^2y + 8xy + 4y^3 = 0 \iff y(x^2+y^2+2x) = 0 \end{cases}$$

The first case holds for the pairs $(-1, 0)$ and $(0, 0)$, which automatically works for case 2. Therefore, $f(z)$ is differentiable at $z = -1, z = 0$. We find that $f'(-1) = f'(0) = 0$.

Example 10: Find the set of points in which $f(z) = \frac{1}{\bar{z}-i}$ is differentiable. If such points exist, find their derivatives.

Apply the same test from Example 9. In this case, we find that only $z = -i$. However, f is not continuous, and consequently, not differentiable at that point! So, $f(z)$ is nowhere differentiable.

3 Elementary Functions

Definition 3.1: Exponential Functions and Properties

An exponential function in the complex $f(z) = e^z$ satisfies the following properties:
(1) $f(z)$ is entire (holomorphic on the complex plane)
(2) e^z is $2\pi i$ periodic (i.e. $e^{3\pi i} = e^{\pi i}$).
(3) $|e^z| = e^x$ for $z = x + iy$.
(4) All of the exponential laws work exactly the same with real numbers.

Verifying (3) and (4) requires us to rewrite $e^z \to e^{x+iy}$ and use Euler's formula to show that they work.

Example 11: Solve $e^z = 1 + i$.

Converting to polar form, $1 + i = \sqrt{2}e^{\frac{i\pi}{4}}$

$$\sqrt{2}e^{\frac{i\pi}{4}} = e^{\ln\sqrt{2}}e^{\frac{i\pi}{4}} = e^{\ln\sqrt{2}+\frac{i\pi}{4}} = e^z \implies z = \frac{1}{2}\ln 2 + \frac{i\pi}{4} + 2\pi ik \text{ for some } k \in \mathbb{Z}.$$

We use the periodicity of e^z to add in the $2\pi ik$ term.

Definition 3.2: Logarithmic Functions

Write z in polar form at $z = re^{i\theta}$ with $\theta = \text{Arg}z \in [-\pi, \pi]$ and $z \neq 0$. The *principal logarithm* of z is

$$\text{Log}z := \ln r + i\theta = \ln|z| + i\text{Arg}z$$

If we choose another argument *argz*, obtain another logarithm

$$\log z = \ln|z| + i\arg z$$

Be careful with notation! Uppercase means we are using the principal logarithm & argument.

Example 12: $\text{Log}(-1-i) = \text{Log}\left(\sqrt{2}e^{-\frac{3\pi i}{4}}\right) = \frac{1}{2}\ln 2 - \frac{3\pi i}{4}$ where $\arg z \in [-\pi, \pi]$.

Example 13: This time, we find $\log(-1-i)$.

$$\log(-1-i) = \frac{1}{2}\ln 2 - \frac{3\pi i}{4} + 2\pi ik, k \in \mathbb{Z}$$

Here we let z be the argument $\arg z \in [-\pi \pm 2\pi k, \pi \pm 2\pi k]$, where k is any whole number.

Examples 12 and 13 stress the importance of notation of log versus Log: using log yields a set, including the principal logarithm, whereas Log only accepts one value: the principal logarithm.

Definition 3.3: Properties of Logarithmic Functions

Compared to exponential functions, the properties of exponential are a little bit more subtle. Let $z, w \in \mathbb{C}$, and $n \in \mathbb{N}$. Then,
(1) $\log(zw) = \log z + \log w + 2\pi i k, k \in \mathbb{Z}$
(2) $\log\left(\frac{z}{w}\right) = \log z = \log w + 2\pi i k, k \in \mathbb{Z}$
(3) $\text{Log} z^n = n\text{Log} z + 2\pi k i$ for some integer k with $|k| \leq \frac{n}{2}$.
Applying properties (1) and (2) to principal logarithms require $k = -1, 0$, or 1.

Example 14: Let $z = w = 4e^{\frac{2\pi i}{3}}$. Compute $\log(zw)$ and $\text{Log}(zw)$ and compare the results.

$$\log(zw) = \log\left(16e^{\frac{4\pi i}{3}}\right) = 2\ln 4 + \frac{4\pi i}{3} + 2\pi i k, k \in \mathbb{Z}$$

which matches the computation for $\log z + \log w$.

$$\text{Log}(zw) = \text{Log}\left(16e^{\frac{4\pi i}{3}}\right) = \text{Log}\left(16e^{-\frac{2\pi i}{3}}\right) = \left(2\ln 4 + \frac{4\pi i}{3}\right) - 2\pi i = 2\ln 4 - \frac{2\pi i}{3}$$

Notice how we must account for the principal argument in the second computation. Therefore, our results are different.

Example 15: Let $z = -1 + i$. Compute $\text{Log} z^2$ and show $\text{Log} z^2 \neq 2\text{Log} z$.

$$\text{Log} z^2 = \text{Log}\left(\left(\sqrt{2}e^{\frac{3\pi}{4}}\right)^2\right) = \text{Log}\left(2e^{\frac{3\pi i}{2}}\right) = \ln 2 - \frac{\pi i}{2} \text{ by principal argument.}$$

$$2\text{Log} z = 2\ln 2 + \frac{3\pi i}{2} \implies \text{Log} z^2 = 2\text{Log} z - 2\pi i$$

This aligns with the property mentioned in the definition above, with $k = -1$.

Definition 3.4: Multi-valued Functions and Branches

A *multi-valued function* can produce multiple distinct output values. The *branch* of a multi-valued function is a single-valued function F on a domain D which is *holomorphic* on D and such that each $F(z)$ is one of the value of $f(z)$. A *branch cut* is the removal of a line or curve l on D in $\mathbb{C}$.

We've seen examples of branches with logarithms, the most common multi-valued function. The *principle branch cut* of a logarithm is a restricted version of the principal logarithm. We can construct other types of branches as well. For example, an α branch cut defines a completely new branch of the logarithm

$$\log z = \ln r + i\theta \text{ where } \theta \in (\alpha - 2\pi, \alpha)$$

For example, choosing $\alpha = -\pi$ yields the same branch cut but **different** branch, compared to the principal branch.

Example 16: Show that the function $f(z) = \text{Log}(z - 2i)$ is holomorphic everywhere except on the portion $x \leq 0$ of the line $y = 2$.

$$\text{Log}(x + i(y-2)) = \ln\left(\sqrt{x^2 + (y-2)^2}\right) + i\text{Arg}(z - 2i)$$

At $y = 2$, this simplifies to $\ln|x| + i\text{Arg}(x - 2i)$. Clearly, it is not differentiable at $x = 0$. However, $\text{Arg}(z - 2i)$ is not differentiable at the branch/cut line $\text{Arg}(z-2i)|_{y=1} = \pi \implies \text{Arg}(x) = \pi \implies x < 0$.

Definition 3.5: Power Functions

For any non-zero z and complex number c, we define the function

$$z^c = e^{c \log z}$$

with its principal value

$$\text{P.V.} z^c = e^{c \text{Log} z}.$$

Example 17: Find the principal value of $(1-i)^{2i}$.

$$\text{Let } f(z) = (1-i)^z = e^{z\text{Log}(1-i)} = e^{z\text{Log}\left(\sqrt{2}e^{-\frac{\pi i}{4}}\right)} = e^{z\left(\frac{1}{2}\ln 2 - \frac{\pi i}{4}\right)}.$$

and so

$$(1-i)^{2i} = e^{2i\left(\frac{1}{2} - \frac{\pi i}{4}\right)} = e^{(\ln 2)i + \frac{\pi}{2}} = e^{\frac{\pi}{2}}(\cos(\ln 2) + i\sin(\ln 2))$$

Example 18: The power function z^c is usually multi-valued. However, if $c = m$ is an integer, prove that z^m is single-valued: i.e. it is independent of the branch of the logarithm used in its definition.

Let $c = m$ be an integer and $\log z$ be any branch of the logarithm such that z is not on the branch cut. Then,

$$z^m = e^{m(\text{Log} z + 2\pi n i)} = e^{m\text{Log} z}e^{2\pi m n i} = e^{m\text{Log} z} \text{ (by periodicity property) } = \text{P.V.} z^m$$

Definition 3.6: Trigonometric Functions

For any $z \in \mathbb{C}$, define

$$\cos z = \frac{e^{iz} + e^{-iz}}{2}, \sin z = \frac{e^{iz} - e^{-iz}}{2i}$$

Example 19: Compute $\cos^{-1}(\sqrt{2})$ with respect to the complex numbers.

This is equivalent to saying that we can find a $z \in \mathbb{C}$ such that $\cos z = \sqrt{2}$.

$$\sqrt{2} = \frac{e^{iz} + e^{-iz}}{2} \implies 2\sqrt{2}e^{iz} + \left(e^{iz}\right)^2 + 1 = 0$$

We multiplied every term by e^{iz} to easily obtain a quadratic equation. Let $u = e^{iz}$, then

$$u^2 + 2\sqrt{2} + 1 = 0 \iff u = \frac{-2\sqrt{2} \pm 4}{2} = -\sqrt{2} \pm 1$$

Setting $u = e^{iz}$ gives

$$z = \frac{1}{i}\log\left(-\sqrt{2} \pm 1\right) = \frac{1}{i}\left(\log\left((\sqrt{2} \pm 1)e^{i\pi}\right)\right) = \frac{1}{i}(\ln\left(\sqrt{2} \pm 1\right) + i\pi) = \frac{1}{i}\ln\left(\sqrt{2} \pm 1\right) + \pi + 2\pi n$$

4 Integration

Definition 4.1: Standard Parametrization of a Path

(1) Parametrization of a Line
Let z_1, z_2 be points of a linear path such that the starting point and end point is z_1 and z_2, respectively. Then, the standard parametrization of the path joining z_1 and z_2 is

$$z(t) = (1-t)z_1 + tz_2, 0 \leq t \leq 1$$

(2) Parameterization of a Circle
The parameterization of a circle with radius R is given as

$$z(t) = Re^{it}, 0 \leq t \leq 2\pi \text{ if counterclockwise }, z(t) = Re^{-it} \text{ if clockwise}$$

If the circular path contains a branch cut $\theta = \alpha$, then we parameterize from $\alpha - 2\pi \leq t \leq \alpha$.

Example 20: Find a parametrization for the line joining $z = 1$ to $z = -1 + 2i$

$$z(t) = (1-t) + (-1+2i)t$$

Example 21: Find a parametrization for the half and full unit circle

$$z_S(t) = e^{it}, 0 \leq t \leq \pi, z_C(t) = e^{it}, 0 \leq t \leq 2\pi$$

Definition 4.2: Contour Integrals

Let $z(t)$ be a parameterization of a contour C from a to b. Then,

$$\int_C f(z)dz = \int_a^b f(z(t))z'(t)dt$$

If C is closed, simple, and positively oriented, we can denote as $\oint_C f(z)dz$.

Example 22: Find $\int_C z^2 dz$ where C is the line from Example 20.

$$f(z(t)) = (1-2t+2it)^2, z'(t) = 2i-2 \implies \int_C z^2 dz = \int_0^1 (1-2t+2it)^2(2i-2)dt = \frac{10}{3} - \frac{2i}{3}$$

Example 23: Compute $\int_C (\overline{z} + i)dz$,where C is the line joining $1+i$ to $-2-i$.

$$z(t) = (1-t)(1+i) + t(-2-i) = 1 - 3t - i - 2ti, z'(t) = -3 - 2i$$

Therefore the integral is computed accordingly

$$\int_C (\overline{z}+i)dz = \int_0^1 f(z(t))z'(t)dt = -\frac{(3+2i)(-1+2i)}{2} = -\frac{7}{2}+2i$$

Aside: A future theorem implies that all integrals with $\overline{z}$ in the integrand must be computed with parametrization! The underlying idea involves $\overline{z}$ having no anti-derivative.

Example 24: Compute $\int_{C_1} \frac{dz}{z}$ and $\int_{C_2} \frac{dz}{z}$ where C_1 is the semi-circle of radius 1 oriented counterclockwise and C_2 is the semi-circle of radius 1 oriented clockwise, both starting at $z=1$.

$$C_1 : z = e^{it}, 0 \le t \le \pi \Longrightarrow \int_{C_1} \frac{dz}{z} = \int_0^{\pi} f(z(t))z'(t) = \int_0^{\pi} \frac{1}{e^{it}} ie^{it} dt = i\pi$$

$$C_2 : z = e^{-it}, 0 \le t \le \pi \Longrightarrow \int_{C_2} \frac{dz}{z} = -i\pi$$

The upcoming definitions and theorems will help us generalize the following result: The value for $\int_C \frac{dz}{z}$ will vary by at most $2\pi i$ depending on the choice of path C. Until then, we visit an example of how we may have to rethink our parametrization.

Example 25: Compute $\oint_C z^i dz$ where we use the principal value and C is the unit circle.

Recall that z^i is a multi-valued function! The problem requires us to work around the principal branch cut along the negative real axis. We cannot use the standard parametrization from (4.2), but instead set $\alpha = \pi$. Then, we obtain a new parametrization

$$z(t) = e^{it}, -\pi \le t \le \pi \Longrightarrow \oint_C z^i dz = \oint_C e^{i\text{Log}z} dz = \int_{-\pi}^{\pi} e^{i-it} ie^{it} dt = \int_{-\pi}^{\pi} ie^{(i-1)t} dt$$

$$= \frac{i}{i-1} e^{(i-1)t}\Big|_{-\pi}^{\pi} = \frac{1}{2}(1-i)\left(e^{\pi} - e^{-\pi}\right)$$

Theorem 4.3: Integrals with Path Independence

Let f defined on a domain $D \subseteq \mathbb{C}$ be given, and let C be a curve. We say $\int_C f$ is *path-independent* if its value only changes on the endpoint of C. **Every** contour integral $\int_C f(z)dz$ over a contour in D is path-independent if and only if $\int_C f(z)dz = 0$ around **every** closed contour. Or,

$$\text{Every } \int_C f \text{ is path independent} \iff \text{Every } \oint_C f = 0$$

The results of this theorem helps us generalize contour integrals between two points for any given path! One of the most properties used in the proof of this theorem states that for any two contours such that $C = C_1 \cup C_2$,
$\int_{C_1} f(z)dz + \int_{C_2} f(z)dz = \int_{C_1} f(z)dz - \int_{-C_2} f(z)dz$. In other words, flipping the direction of the path makes the contour integral negative!
Example 26: Let C_1, C_2 be the contours in Example 24. We have that

$$\int_{C_1} z^2 dz = \int_{C_2} z^2 dz = -\frac{2}{3}$$

This is a nice inference from the theorem. In fact, any such C will give the same value for $\int_C z^2$. While we can argue that this is the case because the contour integral is path-independent regardless of C, we need the following theorem to prove this.

Theorem 4.4: Fundamental Theorem of Calculus

Suppose f is continuous on an open domain D. Then,

$$f \text{ has an anti-derivative on } D \iff \text{All } \int_C f \text{ are path-independent}$$

In such a case, if $F'(z) = f(z)$, then $\int_f = F(z_1) - F(z_0)$.

The largest takeaway from the Fundamental Theorem of Calculus is that: as long as f has an anti-derivative, we no longer need to parametrize C!

Example 27: Evaluate $\int_C z^3 dz$, where C is the line joining $1+i$ to $1-i$.

Because $f(z)$ clearly has an anti-derivative, the FTC states that we can simply compute the anti-derivative at our endpoints

$$\int_C z^3 dz = \frac{1}{4} z^4 \Big|_{1+i}^{1-i} = \frac{1}{4}\left((1-i)^4 - (1+i)^4)\right) = \frac{1}{4}\left(8e^{i\pi} - 8e^{-i\pi}\right) = 0$$

Example 28: Explain why the above theorem does not apply to $\int_C \overline{z}$.

We cannot simply define an anti-derivative for $\overline{z}$ because it is not holomorphic! Therefore, we **are required to** parametrize C.

Example 29: Define a domain D such that $\int_C \frac{dz}{z}$ is guaranteed to be path independent.

This is equivalent to asking: where does f has an anti-derivative $\implies$ where is it holomorphic? We know that $F(z) = \log z + c$, so we must take branch cuts into account. If we let D be the domain of the complex numbers with exclusion of the branch cut, then we gain path independence for all C.

Theorem 4.5: Integral Estimation

(1) Suppose $w : [a, b] \to \mathbb{C}$ is piecewise continuous. Then,

$$\left| \int_a^b w(t)dt \right| \leq \int_a^b |w(t)|dt$$

(2) Suppose C is a contour with length L and let f be piecewise continuous on C. Then $|f(z)|$ is bounded by some $M \geq 0$ on C, and

$$\left| \int_C f(z)dz \right| \leq ML$$

Example 30: Estimate $\left| \int_C \frac{z+2i}{z^8+1} dz \right|$ where C goes from 2 to $2i$.

We first find the length of C. The Pythagorean Theorem tells us $L = 2\sqrt{2}$. To find such M, we must find an upper bound for $|z + 2i|$ and lower bound for $|z^8 + 1|$. A useful result from C is that the maximum distance from z to C is at either endpoint, so $|z| \leq 2$. The minimum distance can be found by bisecting the origin to C, which is ultimately the distance from 0 to $1 + i$. Therefore, $\sqrt{2} \leq |z| \leq 2$. Having a two-sided inequality is crucial for finding each bound.

For $|z + 2i|$, we use the upper bound of $|z| \implies |z + 2i| \leq |z| + |2i| = 4$.
For $|z^8 + 1|$, we use the lower bound and the reverse triangle inequality:
$|z^8 + 1| \geq ||z^8| - |1|| = 15$.
Hence, $M = \frac{4}{15}$ and

$$\left| \int_C \frac{z + 2i}{z^8 + 1} dz \right| \leq ML = \frac{8\sqrt{2}}{15}$$

Example 31: Let C be the arc of the circle $|z| = 2$ joining 2 to $2 + 2i$. Show $\left| \int_C \frac{z+4}{z^3-1} dz \right| \leq \frac{6\pi}{7}$.

The process is a lot simpler as we are already given $|z|$. The length of C is $\frac{1}{4}(2\pi(2)) = \pi$. The upper bound for $|z + 4|$ is $|z + 4| \leq |z| + 4 = 6$ and the lower bound for $|z^3 - 1|$ is $|z^3 - 1| = |1 - z^3| \geq |1 - |z^3|| = 7$. $ML = \frac{6\pi}{7}$.

Example 32: If C is the straight line joining the origin to $1 + i$, show $\left| \int_C z^3 e^{2iz} dz \right| \leq 4$.

$L = \sqrt{2}$. To find a bound for M, we can parametrize C as $z(t) = (1 + i)t, 0 \leq t \leq 1$. So,

$$\left| z^3 e^{2iz} \right| = \left| (1 + i)^3 t^3 e^{2i(1+i)t} \right| = \sqrt{8} \left| t^3 \right| \left| e^{2it} \right| \left| e^{-2t} \right| \leq \sqrt{8}$$

Since $e^{-2t} \leq 1$ and $|e^{2it}| \leq 1$ using Euler's Formula. Therefore, $ML = \sqrt{8} \cdot \sqrt{2} = 4$.

Example 33: If C is the boundary of the triangle with vertices 0, $3i$, and -4, prove that $\left|\oint_C (e^z - \overline{z})dz\right| \leq 60$.

$L =$ perimeter of triangle $= 12$. To find M, we find a bound for $|e^z - \overline{z}|$. We have that $|e^z| = |e^x|$ and $\frac{1}{e^4} \leq |e^x| \leq 1$ and $|\overline{z}| = |z| \leq 4$ is the maximum distance from z to any point on C. So, $M = 1 + 4 = 5$. We thus obtain $ML = 60$.

Theorem 4.6: Cauchy-Goursat

(1) Suppose C is a closed contour in a simply-connected region D. If f is holomorphic on D, then $\int_C f(z)dz = 0$.
(2) Now, suppose C is a simple closed contour, oriented *counter-clockwise*. Let $C_1, ..., C_k$ be non-intersecting simple closed contours in the interior of C, oriented clockwise. If $f(z)$ is holomorphic on the region between and including C and the interior boundaries $C_1, ..., C_k$, then

$$\int_C f(z)dz + \sum_{j=1}^{k} \int_{C_j} f(z)dz = 0$$

We are told that if f is holomorphic on and inside C, then its integral is 0.

Example 34: Explain why for each $f(z)$ given below, $\oint_C f(z)dz = 0$ when the contour C is the unit circle $|z| = 1$.
(a) $f(z) = \frac{z^2}{z+3}$: We have a discontinuity at $z = -3$, which certaintly does not lie inside or on C.
(b) $f(z) = ze^{-z}$: f is entire!
(c) $f(z) = \text{Log}(z + 2)$: f is holomorphic on all points except for the branch cut:
$\text{Re}z \leq -2, \text{Im}z = 0$. The branch cut lies entirely outside of C, so f is holomorphic in C.

By the Cauchy-Goursat Theorem, we can conclude that $\oint_C f(z)dz = 0$ for all three choices of $f(z)$.

Corollary 4.7: Nested Contours

Suppose C_1, C_2 are non-intersecting positively oriented simple closed contours. If f is holomorphic on the region between and including the curves, then

$$\oint_{C_1} f(z)dz = \oint_{C_2} f(z)dz$$

Example 35: Let C_1 be the square with sides $x = \pm 1, y = \pm 1$, and C_2 the circle $|z| = 4$. Explain why $\oint_{C_1} \frac{1}{3z^2+1}dz = \oint_{C_2} \frac{1}{3z^2+1}dz$.

C_1 is entirely contained in C_2, therefore they are non-intersecting. f is holomorphic except at the points $z = \pm\frac{1}{\sqrt{3}}$, interior to both C_1, C_2. Therefore, f is holomorphic in the space **between** C_1 and C_2, so their closed contour integrals are equal.

Theorem 4.8: Cauchy-Integral Formula

Suppose f is holomorphic everywhere on and inside a simple closed contour C. If z_0 is any point inside C, then

$$f(z_0) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z - z_0}dz$$

More generally, if f is infinitely differentiable at z_0 with n^{th} derivative

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\oint_C \frac{f(z)}{(z - z_0)^{n+1}}dz$$

To generalize, if $f(z)$ is holomorphic everywhere in C, then its contour integral is 0 by Cauchy-Goursat. If it is holomorphic except for a point z_0 in $\mathbb{C}$, we must apply the Cauchy-Integral formula by and show

$$\oint_C \frac{f(z)}{(z - z_0)^{n+1}}dz = \frac{2\pi i f^{(n)}(z_0)}{n!}$$

Example 36: Let C be the square with $x = 0, 1$ and $y = 0, 1$. Evaluate the integral $\oint_C \frac{1}{z-a}dz$ when
(a) a is *exterior* to the square: The closed contour is zero by the Cauchy-Goursat Theorem.
(b) a is *interior* to the square: We have to use the Cauchy-Integral Formula. Here, $f(z) = 1$ and we evaluate $f(a)$

$$\oint_C \frac{1}{z - a}dz = 2\pi i f(a) = 2\pi i.$$

Example 37: Let C denote the boundary of the square with sides $x = \pm 2, y = \pm 2$. Evaluate $\oint_C \frac{e^z + e^{-z}}{z(z^2+16)}$.

We have that $z(z^2 + 16) = 0$ when $z = 0, z = \pm 4i$, for which $z = 0$ is inside C. Therefore we evaluate $f(0)$ where $f(z) = \frac{e^z + e^{-z}}{z^2+16}$. Therefore,

$$\oint_C \frac{e^z + e^{-z}}{z(z^2 + 16)} = 2\pi i f(0) = 2\pi i \cdot \frac{1}{16} = \frac{\pi i}{8}.$$

Example 38: Let C denote the circle of radius 3 centered at $z = -i$. Evaluate $\oint_C \frac{e^z}{(z^2-4iz-3)^2}dz$.

Factoring the denominator gives $(z-i)(z-3i) \implies z = i, 3i$. Because $z = i$ lies inside C, we compute $f'(i)$, because we have a squared term.

$$f(z) = \frac{e^z}{(z-3i)^2} \implies f'(z) = \frac{e^z((z-3i)^2 - 2(z-3i))}{(z-3i)^4} \implies f'(i) = \frac{1}{64}e^i(4-i)$$

Therefore
$$\oint_C \frac{e^z}{(z^2-4iz-3)^2} dz = 2\pi i f'(i) = \frac{1}{32}e^i \pi(1+4i)$$
where $e^i = \cos 1 + i \sin 1$.

Lemma 4.9: Cauchy's Inequality

If f is holomorphic on and inside the circle C of radius R centered at z_0 and $|f(z)| \leq M$ on C, then
$$\left|f^{(n)}(z_0)\right| = \frac{n!}{2\pi}\left|\oint_c \frac{f(z)}{(z-z_0)^{n+1}} dz\right| \leq \frac{n!M}{R^n}.$$

One important application of this lemma is that we can prove interesting qualities about entire and bounded functions!

Example 39: Suppose f is entire and that $|f(z)| < c|z|$ for some constant $c \in \mathbb{R}^+$. Prove that $f(z) = kz$ where $k \in \mathbb{C}$ satisfies $|k| \leq c$.

Idea: Use Cauchy's Inequality. If we let C_R be the disk of radius R such that $|z - z_0| \leq R$ $\forall z_0 \in \mathbb{C}$, we can apply Cauchy's Inequality as such (since f entire $\implies$ f is holomorphic on the disk): Since $|z| = |z_0| + R$,
$$|f''(z_0)| \leq \frac{2c(|z_0|+R)}{R^2} \to 0 \text{ as } R \to \infty.$$
Here $n = 2$ and $M = c(|z_0| + R)$. Therefore, $f''(z) = 0 \; \forall z \in \mathbb{C}$, implying that f is at most a linear polynomial $f(z) = kz$, where $|f(z)| \leq c|z| \implies |k| \leq c$. In fact, we can extend this to all linear polynomials, if $|f(z)| \leq |cz + d|$, then f is either constant or linear in z!

Theorem 4.10: Liouville's Theorem

Every bounded, entire function is constant.

Example 40: Recall $f(z) = \sin z$. If z only consists of the real numbers, we can say f is bounded by $[-1, 1]$. However, for $z \in \mathbb{C}$, $\sin z$ is a linear combination of e^z and e^{-z}, which are unbounded functions. Therefore, Liouville's thoerem implies our intuition that $\sin z$ is non-constant.

Corollary 4.11: Fundamental Theorem of Algebra

Every non-constant polynomial has a root in $\mathbb{C}$.

Theorem 4.12: Maximum Modulus Principle

Suppose f is holomorphic and non-constant on a connected, open domain D. Then $|f(z)|$ has no maximum value on D.

Example 41: Let $f(z) = 2z^2 + i$ on the upper semi-disk D with radius 1. Find the maximum and minimum values on D.

We decompose the boundary into two parts:

- The curve along $r = 1$: It is easier to verify this by writing f in polar form: $f(z) = (\sqrt{2}e^{i\theta})^2 + 1 = 2e^{2i\theta} + 1$, and so $|f(z)| = \sqrt{(2\cos(2\theta))^2 + (2\sin(2\theta) + 1)^2} = \sqrt{5 + 4\sin(2\theta)}$. It is straightforward to verify that the maximum and minimum occur when $\theta = \frac{\pi}{4}$ and $\theta = -\frac{\pi}{4}$, corresponding to values 3 and 1, respectively.

- Along $y = 0, |x| \leq 1$. $|f(z)| = \sqrt{4x^4 + 1}$. Because f is strictly increasing, the maximum and minimum are $\sqrt{5}$ and 1, respectively.

Therefore the maximum and minimum values of f on D are 3 and 1, respectively.

Example 42: Let $f(z)$ be a *non-zero* holomorphic function on a closed bounded domain. By considering $g(z) = \frac{1}{f(z)}$, show that the minimum value of $|f(z)|$ also occurs on the boundary.

We have that $g(z)$ is also holomorphic on D. Also, we have that $g(z)$ is non-constant (because f is not entire!). Therefore, by the Maximum Modulus Principle, $|g(z)|$ has no maximum in D, but rather on D. Therefore, $|f(z)| = \frac{1}{|g(z)|}$ has its minimum when g has its maximum, which also occurs on the boundary.

Example 43: Consider $f(z) = \exp\left(-|z|^2\right)$ defined on the unit disk $|z| \leq 1$. What is its maximum modulus, and where is it found? Why doesn't this contradict the maximum modulus principle?

We have that $|f(z)| = f(z)$, and since $|f(z)|$ is strictly decreasing, the maximum value is obtained at $z = 0$, which is *inside* of the disk! However, this does not contradict the Maximum Modulus Principle because $f(z)$ is *not* holomorphic on D for it depends on a non-holomorphic function $\overline{z}$ (Remember we can write $|z|^2 = z\overline{z}$). Therefore, f is not holomorphic $\implies |f(z)|$ is not guaranteed to have its maximum value on D.

5 Series and Analytic Functions

Before diving into important results of series in complex analysis, we introduce this section with some (hopefully!) familiar results.

Definition 5.1: Infinite Series and Convergence

The n^{th} *partial sum* of a sequence $(z_n)_{n=0}^{\infty}$ is the complex number

$$s_n = \sum_{k=0}^{n} z_k = z_0 + ... + z_n$$

The (infinite) series $\sum z_n$ converges if (s_n) converges, and diverges otherwise. We obtain *absolute convergence* if $\sum |z_n|$ converges and *conditional convergence* if it converges but *not absolutely.*

These statements are exactly the same as in real analysis. In addition, the general rules and tests also apply:

Theorem 5.2: Basic Results of Series

The following results from series with real numbers also apply to complex numbers

- Linearity between multiple series
- Common convergence tests (i.e. ratio, root, comparison)

Now, we introduce the definition power series, and generalize this definition onto disks.

Definition 5.3: Power Series and Analytic Functions

A *power series centered at* z_0 is the function

$$p(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n$$

where $z_0, a_n \in \mathbb{C}$ are coefficients. In addition, we say $f\colon D \to \mathbb{C}$ is analytic if every $z_0 \in D$ has a neighborhood on which $f(z)$ equals a power series entered at z_0. More precisely, to be analytic at a point z_0 is to be analytic on some neighborhood of z_0. The *radius*, or *disk of convergence* is expressed as $R_0 := \sup\{|z - z_0 : p(z) \text{ converges}\}$, or the maximum radius of the disk in which f converges to its power series.

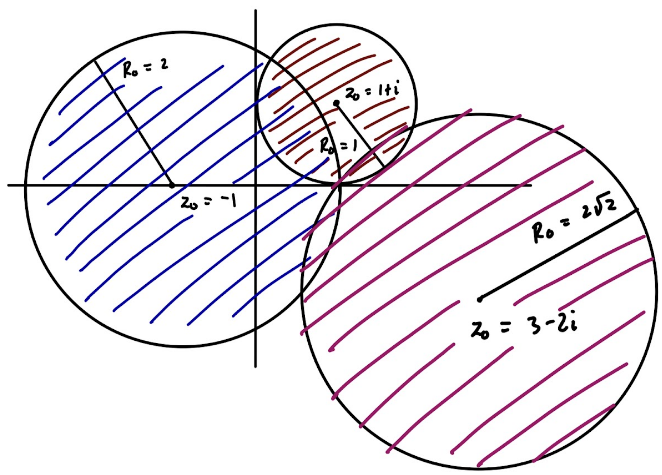
We will discuss analytic functions later, but the important result is that analytic functions, for each z_0, will equal a power series centered at z_0, with unique coefficients.

Example 44: Derive a power series representation for $f(z) = \frac{1}{1-z}$. Find and sketch the disks of convergence corresponding to the centers $z_0 = -1, 1+i, 3-2i$.

We recall the formula for a geometric series. If $|z| < 1$, then $\sum z^n = \frac{1}{1-z}$. Let $z_0 \neq 1$, then

$$f(z) = \frac{1}{1-z} = \frac{1}{1-z_0-(z-z_0)} = \frac{1}{1-z_0}\left[\frac{1}{1-\frac{z-z_0}{1-z_0}}\right] = \frac{1}{1-z_0}\sum_{n=0}^{\infty}\left(\frac{z-z_0}{1-z_0}\right)^n$$

Recall that we require $\left|\frac{z-z_0}{1-z_0}\right| < 1 \iff |z-z_0| < |1-z_0|$. Therefore, $f(z_0)$ converges to a power series centered at z_0 with radius $|1-z_0|$, provided $z_0 \neq 1$. The disks of radius for the desired values of z_0 are drawn below:



Example 45: Apply the same idea from Example 44 to find a power series centered at $z_0 \neq i$ which equals the function $g(z) = \frac{2}{1+iz}$. What is its radius of convergence?

We can follow the same computation in Example 44, which would lead us to the following result: Suppose $z_0 \neq i$. Then, $g(z_0)$ converges to the power series

$$\frac{2}{1+iz_0}\sum_{n=0}^{\infty}\left(\frac{(-i)(z-z_0)}{1+iz_0}\right)^n$$

centered at z_0, with radius of convergence $R_0 = |1+iz_0|$.

Example 46: Find all values of z for which the series $\sum \frac{1+i}{(n+1)(4+3i)^n}(z-1+2i)^n$ converges.

To help simplify the expression, let w be a function of z such that $w = \frac{z-1+2i}{4+3i}$. Then, we can rewrite the series as

$$(1+i)\sum \frac{w^n}{n+1}$$

We want the series to converge, so we want $|w| < 1$. This implies

$$\left|\frac{z-1+2i}{4+3i}\right| < 1 \implies |z-1+2i| < |4+3i| = 5 \implies |z-1+2i| < 5.$$

Therefore, we have a power series centered at $z_0 = 1-2i$ with radius of convergence $R_0 = 5$.

Theorem 5.4: Taylor's Theorem

Suppose $f(z)$ is infinitely differentiable at z_0. Its *Taylor Series* about z_0 is the power series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z-z_0)^n$$

If $f(z)$ is holomorphic on a disk $|z-z_0| < R$, then $f(z)$ equals its Taylor Series with no error term. Or,

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z-z_0)^n \text{ on the disk.}$$

Example 47: Write the Taylor Series for $f(z) = e^z$ about $z_0 = 0, z_0 = i\pi$. What can be said about f?

About $z_0 = 0$, we call such series the *Maclaurin Series*. Recall that

$$\sum_{n=0}^{\infty} \frac{z^n}{n!}.$$

About $z_0 = i\pi$, we find that $f^{(n)}(z_0) = e^{i\pi} = -1$. Therefore, we obtain the following Taylor Series

$$\sum_{n=0}^{\infty} -\frac{1}{n!}(z-i\pi)^n.$$

Because e^z is entire, f will equal its Taylor Series, and in fact, everywhere!

Example 48: Find the Taylor Series of $\cos z$ centered about $z_0 = i$.

We first compute the Taylor coefficients. Treat two separate cases (as the derivative of $\pm\cos z$ alternate between $\pm\sin z$ and itself) accordingly:

$$f^{(2n)}(i) = (-1)^n \cos i = (-1)^n \frac{e^{i^2} + e^{-i^2}}{2} = (-1)^n \frac{e + e^{-1}}{2}$$

$$f^{(2n+1)}(i) = (-1)^n \sin i = (-1)^n \frac{e^{i^2} - e^{-i^2}}{2i} = (-1)^n i \frac{e - e^{-1}}{2}$$

Combining both coefficients we obtain

$$\cos(z - i) = \sum_{n=0}^{\infty} \frac{(-1)^n (e + e^{-1})}{2(2n)!}(z - i)^{2n} + \frac{(-1)^n (e - e^{-1})}{2(2n+1)!} i(z - i)^{2n+1}$$

In fact, we can take out the constant terms, and find that

$$\cos(z - i) = \cos i \cos z + \sin i \sin z$$

which is the additive identity for cosine!

Example 49: Consider $f(z) = \frac{1}{z}$. For any $z_0 \neq 0$, find the Taylor Series of $f(z)$ about z_0. What is its disk of convergence?

The derivatives of f at z_0 are

$$f'(z_0) = -\frac{1}{z_0^2}, \quad f''(z_0) = \frac{2}{z_0^3}, \quad \ldots, \quad f^{(n)}(z_0) = \frac{(-1)^n n!}{z_0^{n+1}}$$

Hence

$$f(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{z_0^{n+1}}(z - z_0)^n = \frac{1}{z_0}\sum_{n=0}^{\infty}(z - z_0)^n = \frac{1}{z_0}\sum_{n=0}^{\infty} \frac{(-1)^n}{z_0^n}(z - z_0)^n$$

As for the disk of convergence, we want $\left|\frac{z - z_0}{z_0}\right| < 1$, implying that $|z - z_0| < |z_0|$ where f converges, except for when $z_0 = 0$.

Corollary 5.5: Holomorphic and Analytic Functions (Pt. 1)

Every holomorphic function is analytic.

We will observe the converse later.

Example 50: Consider $f(z) = \text{Log}z, D = \mathbb{C}\backslash\{\text{non-pos x-axis}\}$. Derive its Taylor Series centered at $z_0 = i$ and identify its disk of convergence.

$$f(i) = \text{Log}i = \frac{i\pi}{2}, f^{(n)}(i) = \frac{(-1)^{n-1}(n-1)!}{i^n}$$
$$\implies f(z) = \text{Log}z = \frac{i\pi}{2} + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{ni^n}(z-i)^n$$

On the disk $|z - i| < 1$, Log$z$ will equal its Taylor Series. On the boundary circle, f will not converge when $z = 0$ (because of the branch cut!).

Definition 5.6: Uniform Convergence

Suppose $f(z) = \sum a_n(z - z_0)^n$ is a power series with n^{th} partial sum $s_n(z)$ and remainder $\rho_n(z) = f(z) - s_n(z)$. We say that the series *converges uniformly* on a domain D if

$$\forall \epsilon > 0, \exists N \text{ such that } n > N, z \in D \implies |\rho_n(z)| < \varepsilon.$$

If R_0, R_1 are radii of convergence of a power series centered at z_0 such that $R_1 < R_0$, then the series converges uniformly on the closed disk $|z - z_0| \leq R_1$.

The notion of uniform convergence is crucial in establishing equivalence between holomorphic and analytic functions, as we will see in the upcoming theorems/corollaries:

Theorem 5.7: Term-by-term Integration

Suppose $f(z) = \sum a_n(z-z_0)^n$ has radius convergence R_0. Let $g(z)$ on some contour C in the open disk of convergence $|z - z_0| < R_0$. Then, we may integrate term-by-term:

$$\int_C g(z)f(z)dz = \sum_{n=0}^{\infty} a_n \int_C g(z)(z - z_0)^n dz$$

As for the proof, we revisit the ML-inequality and use the definition of uniform convergence (since C is compact) to control the size of $\left|\int_C g(z)f(z)dz - \sum_{k=0}^{n} a_k \int_C g(z)(z - z_0)^k dz\right|$.

By choosing such $g(z)$, we now show equivalence between holomorphic and analytic functions, as well as proving unique representation for analytic functions.

Corollary 5.8: Holomorphic and Analytic Functions (Pt. 2)

Suppose $f(z) = \sum a_n(z-z_0)^n$ has positive radius of convergence R_0. We establish the following:

(1) *Term-by-term integration*: Fix $g(z) = 1$. Then,

$$\int_C f(z)dz = \sum_{n=0}^{\infty} a_n \int_C (z-z_0)^n dz = \sum_{n=0}^{\infty} \frac{a_n}{n+1}(z-z_0)^{n+1}\Bigg|_{C(\text{end})}^{C(\text{start})}$$

(2) *Holomorphicity*: We see by (1) that $\int_C f$ is *path-independent* for any contour in the open disk of convergence, implying that f is holomorphic on that disk. So, all analytic functions are holomorphic.

(3) *Term-by-term differentiation*: Define $g(z) = \frac{1}{2\pi i(z-w)^2}$ for $|w-z_0| < R_0$. A simple application of Cauchy's Integral Formula (4.8) to compute $f'(w)$ suffices.

(4) *Unique representation*: The power series $\sum a_n(z-z_0)^n$ is indeed the Taylor Series of $f(z)$, with unique coefficients $a_n = \frac{f^{(n)}(z_0}{n}$.

The conclusions of this corollary are significant:

- Since analyticity and holomorphicity are equivalent now, we will from now on refer to holomorphic functions as analytic
- We can now compute Taylor and Maclaurin series through factoring, differentiation, and integration
- Regardless of how we obtain the series we want, the function is guaranteed to equal said series!

Example 51: Find a power series representation and the radius of convergence for $f(z) = \frac{z}{3-z}$ about $z_0 = 0$.

We want to manipulate f to obtain a similar form $\frac{1}{1-u}$. This can be obtain by factoring out 3 from the denominator.

$$\frac{z}{3-z} = \frac{z}{3}\left(\frac{1}{1-\frac{z}{3}}\right) = \frac{z}{3}\sum_{n=0}^{\infty}\left(\frac{z}{3}\right)^n = \sum_{n=0}^{\infty}\left(\frac{z}{3}\right)^{n+1} \Longrightarrow |z| < 3 \Longrightarrow R_0 = 3.$$

Example 52: Find a power series representation and the radius of convergence for $f(z) = z \sin z^2$ about $z_0 = 0$.

$$f(z) = z\left(\sum_{n=0}^{\infty} \frac{(-1)^n z^{4n+2}}{(2n+1)!}\right) = \sum_{n=0}^{\infty} \frac{(-1)^n z^{4n+3}}{(2n+1)!}$$

A simple application of the ratio test $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$ reveals that $f(z)$ converges to its power series everywhere and $R_0 = \infty$.

Example 53: By expressing $f(z)$ as a Maclaurin series, show that it is entire

$$f(z) = \begin{cases} \frac{1}{z^2}(1 - \cos z) & \text{if } z \neq 0 \\ \frac{1}{2} & \text{if } z = 0 \end{cases}$$

First, expand $\cos z$ into its Maclaurin Series.

$$f(z) = \frac{1}{z^2}\left(1 - \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!}\right)$$

Notice how 1 is the first term in the series. Therefore, we can re-index the series since the first term will cancel out with 1.

$$f(z) = \frac{1}{z^2}\left(\sum_{n=1}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!}\right) = \sum_{n=1}^{\infty} \frac{(-1)^n z^{2n-2}}{(2n!)}$$

If we let $k = n - 1$, then we obtain a Maclaurin series indexed at 0.

$$f(z) = \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k}}{(2k+2)!}, \qquad f(0) = \frac{1}{2}.$$

Therefore, we verify that f equals its Maclaurin series everywhere and so it is entire.

Example 53: Suppose $f(z)$ is analytic and non-constant at z_0. Prove that

$$\exists R > 0 \text{ such that } 0 < |z - z_0| < \epsilon \implies f(z) \neq f(z_0)$$

To what extent can you weaken the hypothesis $f'(z_0) \neq 0$?

Let $f(z)$ be analytic at z_0. Then, it equals its Taylor Series centered around z_0:

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + \frac{f''(z_0)}{2}(z - z_0)^2 + ... + \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n$$

$$\iff f(z) - f(z_0) = (z - z_0)\left(f'(z_0) + \frac{1}{2}f''(z_0)(z - z_0) + ... + \frac{1}{n!}f^{(n)}(z_0)(z - z_0)^{n-1}\right)$$

Let $g(z) = f'(z_0) + \frac{1}{2}f''(z_0)(z-z_0) + ... + \frac{1}{n!}f^{(n)}(z_0)(z-z_0)^{n-1}$ such that $g(z)(z-z_0) = f(z) - f(z_0)$. Because f is non-constant, $f'(z_0) \neq 0 \implies g(z_0) \neq 0$. Since g is analytic, it is also continuous, and so $\exists R > 0$ such that $g(z) \neq 0$ for $|z - z_0| < R$. So, if $0 < z - z_0 < R$, $f(z) - f(z_0) = (z - z_0)g(z)$. Since $z \neq z_0$, and $g(z) \neq 0, f(z) - f(z_0) \neq 0$, as required.

We can weaken the hypothesis by saying that at least one ordered derivative, $f^{(k)}(z_0)$, is nonzero.

Corollary 5.9: Analytic Continuation

Suppose $f(z), g(z)$ are analytic on an open connected domain D, and that $f(z) = g(z)$ on some contour C in D. Then $f(z) = g(z)$ on D.
$(f : D \to \mathbb{C}, g : E \to \mathbb{C}, D \subseteq E)$.

Example 54: Consider the Maclaurin series $f(z) = \sum_{n=0}^{\infty}(-1)^n z^{2n}$ on the disk $|z| < 1$. Show that $h(z) = \frac{1}{z^2+1}$ is the analytic continuation of $f(z)$ to $\mathbb{C}\backslash\{i, -i\}$.

We have that $z^2 + 1 = 0 \iff z = \pm i$. So, h is analytic on $\mathbb{C}\backslash\{i, -i\}$. Now, we find a power series representation for h.

$$\frac{1}{1+z^2} = \frac{1}{1+(-z)^2} = \sum_{n=0}^{\infty}(-1)^n(z^n)^2 = \sum_{n=0}^{\infty}(-1)^n z^{2n}$$

which is the Maclaurin series for f. Therefore, $f(z) = h(z)$ for $|z| < 1$ and h is the analytic continuation of $f(z)$ to $\mathbb{C}$ on $\{-i, i\}$.

Example 55: Find the analytic continuation of $f(z) = \frac{2\cos z^2 - 2 + z^4}{z^8}$ $(D = \mathbb{C}\backslash\{0\})$ to $E = \mathbb{C}$.

Consider $\cos z^2 = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} z^{4n}$ (for all z). Then, on D,

$$f(z) = \frac{2}{z^8}\sum_{n=2}^{\infty}\frac{(-1)^n}{(2n)!}z^{4n} = 2\sum_{n=2}^{\infty}\frac{(-1)^n}{(2n)!}z^{4(n-2)}$$

By re-indexing the series, we found a suitable power series representation for $f(z)$. Therefore, $f(z)$ is analytic and it converges to the power series everywhere, including at zero. Using the definition, we choose

$$g(z) = 2\sum_{n=2}^{\infty}\frac{(-1)^n}{(2n)!}z^{4(n-2)}.$$

Definition 5.10: Zeros of Analytic Functions

Suppose z_0 is a zero of an analytic function $f(z)$.

- We say that z_0 is a *zero of order* $m \geq 1$ if $f^{(m)}(z_0)$ is the first non-zero derivative. A zero of order 1 is also called a *simple zero*.
- If all derivatives are zero, z_0 is a *non-isolated zero*: plainly $f(z) \equiv 0$ on some disk $|z - z_0| < R$.

Theorem 5.11: Order of Zeros Pt. 1

An analytic function $f(z)$ has a zero z_0 of order m if and only if $f(z) = (z - z_0)^m \psi(z)$ where $\psi(z)$ is analytic at z_0 and $\psi(z_0) \neq 0$. Indeed, on some disk $|z - z_0| < R$,

$$f(z) = \sum_{n=m}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n = (z - z_0)^m \psi(z).$$

Theorem 5.12: Isolated Zeros

Suppose z_0 is a zero of an analytic function $f(z)$

- If z_0 has order m, then there exists a punctured disk $0 < |z - z_0| < R$ on which $f(z) \neq 0$. Or, we say z_0 is *isolated*
- If z_0 is non-isolated and the domain D of $f(z)$ is open and connected, then $f(z) \equiv 0$ on D.

6 Laurent Series, Residues, and Poles

We used Taylor Series to find a series representation of an analytic function on a disk. While it is still an incredibly powerful tool, we are not always guaranteed to extract useful information about a function's behavior at certain points. Therefore, we motivate a new type of series whose convergence is *not always* over a disk.

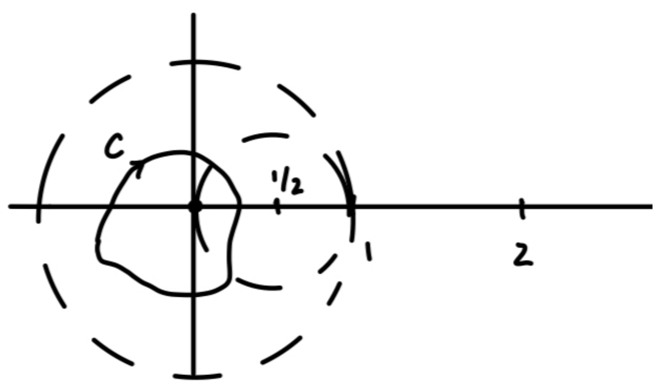
Example 56: Let $f(z) = \frac{1}{z(1-z)}$. If we wanted to find its Taylor Series centered about $z = 0$, we simply multiply the Taylor Series for $\frac{1}{1-z}$ by $\frac{1}{z}$:

$$\frac{1}{z}\sum_{n=0}^{\infty} z^n = \sum_{n=0}^{\infty} z^{n-1}.$$

which is valid on the punctured disk $0 < |z| < 1$. Alternatively, if we wanted to find the Taylor Series centered around $z = \frac{1}{2}$, we can complete the square in the denominator and get

$$f(z) = \frac{1}{z(1-z)} = \frac{1}{-(z-\frac{1}{2})^2 + \frac{1}{4}} = \frac{4}{1-(2z-1)^2} = 4\sum_{n=0}^{\infty}(2z-1)^{2n}.$$

The second series has a much smaller domain compared to what the first series gives. In fact, the first series orbits about $z = 0$, which is important if we think about curves that are also contained within the disk and also orbit the origin. This is the main advantage of the first series and is the result that we will find with series that have *negative terms*. The disks of convergence for both series are shown below



Note that the first series can be re-indexed as

$$\sum_{n=-1}^{\infty} z^n$$

This introduces the coefficient a_{-1} for z^{-1}. Such terms are the foundation of the *Laurent Series*. In general, we can consider a series with infinitely many negative and positive terms.

$$\sum_{n=-\infty}^{\infty} a_n(z-z_0)^n = \sum_{n\geq 0} a_n(z-z_0)^n + \sum_{n\leq -1} a_n(z-z_0)^n$$

If we let $w = (z-z_0)^{-1}$, then

$$\sum_{n=-\infty}^{\infty} a_n(z-z_0)^n = \sum_{n\geq 0} a_n(z-z_0)^n + \sum_{n\geq 1} a_{-n}w^n.$$

Hence the series with infinite terms is really just a sum of two Taylor Series, each with their own radius of convergence R_1, R_2. Naturally, in order for the main series to convergence, it must converge within both radii of convergence. We find that such series will converge on an *annulus*.

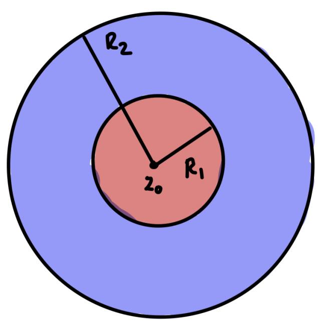
Theorem 6.1: Annulus of Convergence

Let $\sum_{n=-\infty}^{\infty} a_n(z-z_0)^n$ be a series. Then, denote

$$R_1 = \inf\{|z-z_0| : f(z) \text{ converges}\}$$

$$R_2 = \sup\{|z-z_0| : f(z) \text{ converges}\}$$

The series then converges absolutely to a continuous function on the (open) *annulus of convergence* $R_1 < |z-z_0| < R_2$ and uniformly on any subannulus. We have divergence if $|z-z_0| < R_1$ or $|z-z_0| > R_2$.
Similar to power series, convergence must be tested for both boundaries.



From the definition, if $R_1 = 0$, then we reduce to a punctured disk. We can also have $R_2 = \infty$. Combining both ideas $(0 < |z-z_0| < \infty)$ gives us convergence everywhere except for $z_0 = 0$.

Compared with power series, uniform convergence over an annulus lends the same results: the series is continuous, differentiable and integrable term-by-term, and analytic.

Definition 6.2: Laurent Series

Let $f(z)$ be analytic on an annulus $R_1 < |z - z_0| < R_2$ and C be any simple closed curve in the annulus which orbits z_0. The *Laurent Series* of $f(z)$ on this annulus is the series

$$\sum_{n=-\infty}^{\infty} a_n(z-z_0)^n \text{ where } a_n = \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^{n+1}}dz.$$

The terms a_n are the corresponding *Laurent coefficients*.

If $f(z)$ is analytic on the disk $|z - z_0| < R_2$, then the Taylor and Laurent Series are equivalent. This is a rare case; Laurent Series are generally the extrapolation of Taylor Series and equivalence is harder to obtain when dealing with multiple regions.

Example 57: We revisit the function f given in Example 56, which is analytic on the annulus $0 < |z| < 1$. With C as the circle of radius $\frac{1}{2}$ centered at the origin, we can compute its Laurent Series by first rewriting it as $f(z) = \frac{1}{z(1-z)} = \frac{1}{z} + \frac{1}{1-z}$ through partial fraction decomposition. Then, we find a general expression for the coefficients a_n

$$a_n = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z^{n+1}}dz = \frac{1}{2\pi i}\oint_C \frac{1}{z^{n+2}(1-z)}dz$$

The integral can computed using Cauchy's Integral Formula:

$$\frac{1}{(n+1)!}\frac{d^{n+1}}{dz^{n+1}}|_{z=0}(1-z)^{-1} = (1-z)^{-n-1}|_{z=0} = 1 \text{ if } n \geq -1$$

If $n \leq 2$, then $\frac{1}{z^{n+2}(1-z)}$ is analytic on/inside C and so $a_n = 0$. Therefore, the Laurent Series is

$$\sum_{n=-1}^{\infty} z^n$$

We now look at some more examples of finding Laurent Series representations of functions.

Example 58: Find a Laurent series representation for $f(z) = \frac{3}{z^2}e^{2z}$ and compute $\oint_C f(z)dz$ where C is a simple closed curve encircling the origin.
We use the Taylor Series for e^{2z}, then simplify and re-index the series

$$f(z) = \frac{3}{z^2}\sum_{n=0}^{\infty}\frac{(2z)^n}{n!} = \sum_{n=0}^{\infty}\frac{3 \cdot 2^n \cdot z^{n-2}}{n!} = \sum_{n=-2}^{\infty}\frac{3 \cdot 2^n \cdot z^n}{(n+2)!}.$$

Cauchy-Goursat tells us we need only look at when $n = -1$, or the coefficient a_{-1}. So, $\int_C f(z)dz = 2\pi i \cdot 6 = 12\pi i$.

Example 59: Find a Laurent Series centered at $z_0 = 0$ for the function

$$f(z) = \frac{1}{z(z-2i)} = \frac{i}{2}\left(\frac{1}{z} - \frac{1}{z-2i}\right)$$

on the domain $D_1 = \{z : 0 < |z| < 2\}$.

Idea: Extract a power series for $\frac{1}{z-2i}$ as we normally would.

$$\frac{1}{z-2i} = -\frac{1}{2i}\left(\frac{1}{1-\frac{z}{2i}}\right) = \frac{i}{2}\sum_{n=0}^{\infty}\frac{z^n}{(2i)^n} \implies f(z) = \frac{i}{2}\left(\frac{1}{z} + \sum_{n=0}^{\infty}\frac{z^n}{(2i)^n}\right) = \frac{i}{2z} - \sum_{n=0}^{\infty}\frac{z^n}{2^{n+2}i^n}.$$

Example 60: Using the same function f from Example 59, find a Laurent Series centered at $z_0 = 0$ on the domain $D_2 = \{z : |z| > 2\}$

Idea: We want the series to converge on an infinitely large region, so we want to construct a series of only negative powers. This can be obtained by factoring out $\frac{1}{z}$ from $\frac{1}{z-2i}$.

$$\frac{1}{z-2i} = \frac{1}{z}\left(\frac{1}{1-\frac{2i}{z}}\right) = \frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{2i}{z}\right)^n = \sum_{n=0}^{\infty}\frac{(2i)^n}{z^{n+1}}$$

$$\implies f(z) = \frac{i}{2}\left(\frac{1}{z} - \sum_{n=0}^{\infty}\frac{(2i)^n}{z^{n+1}}\right) = \frac{i}{2z} - \sum_{n=0}^{\infty}\frac{2^{n-1}i^{n+1}}{z^{n+1}} = \sum_{n=1}^{\infty}\frac{2^{n-1}i^{n+1}}{z^{n+1}}$$

If we wanted to find a Laurent Series centered at $z_0 = 2i$ on D_1, we would need to rewrite $z(z-2i)$ into the form $\frac{k}{1-(z-2i)^2}$, where $k \in \mathbb{C}$.

Example 61: Find a Laurent Series for $f(z) = \frac{z}{(z-1)(z-3)}$ on the punctured disk $0 < |z-1| < 2$.

Idea: First use partial fractions to obtain the sum $\frac{A}{z-1} + \frac{B}{z-3}$, then rewrite the second fraction to obtain a power series centered about $z = 1$.

$$\frac{z}{(z-1)(z-3)} = \frac{A}{z-1} + \frac{B}{z-3} \implies A(z-3) + B(z-1) = z \implies A = -\frac{1}{2}, B = \frac{3}{2}$$

Now, find the power series representation for $\frac{3}{2}\cdot\frac{1}{z-3}$ about $z = 1$:

$$\frac{3}{2}\cdot\frac{1}{z-3} = \frac{3}{2}\cdot\frac{1}{(z-1)-2} = -\frac{3}{4}\cdot\frac{1}{1-\frac{z-1}{2}} = -\frac{3}{4}\sum_{n=0}^{\infty}\frac{(z-1)^n}{2^n} = -3\sum_{n=0}^{\infty}\frac{(z-1)^n}{2^{n+2}}$$

Therefore,

$$f(z) = -\frac{1}{2(z-1)} - 3\sum_{n=0}^{\infty}\frac{(z-1)^n}{2^{n+2}} \text{ on } 0 < |z-1| < 2.$$

Likewise, if we wanted to find the Laurent Series on $0 < |z-3| < 2$, we find a power series representation for $\frac{1}{z-1}$ about $z = 3$.

Example 62: Let a be a complex number. Show that

$$\frac{a}{z-a} = \sum_{n=1}^{\infty} \frac{a^n}{z^n} \text{ whenever } |a| < |z|.$$

Apply the same idea from Example 60:

$$\frac{a}{z-a} = \frac{a}{z}\left(\frac{1}{1-\frac{a}{z}}\right) = \frac{a}{z}\sum_{n=0}^{\infty}\frac{a^n}{z^n} = \sum_{n=0}^{\infty}\frac{a^{n+1}}{z^{n+1}} = \sum_{n=1}^{\infty}\frac{a^n}{z^n}$$

We want $\left|\frac{a^n}{z^n}\right| < 1$, which implies $|a^n| < |z^n| \iff |a| < |z|$.

After going through a few examples, the following theorem and corollary should feel trivial:

Theorem 6.3: Laurent's Theorem

An analytic function on an open annulus equals its Laurent Series

Corollary 6.4: Properties of Laurent Series

The notions of term-by-term integration/differentiation, analyticity, and uniqueness from power series also apply to Laurent Series.

We now want to further investigate easier ways of computing integrals of analytic function. Let us translate the concept of discontinuities from real analysis to complex analysis with some new vocabulary:

Definition 6.5: Singularities, Residues, and Poles

Say z_0 is an *isolated singularity* of $f(z)$ if the function is analytic on a punctured disk $0 < |z - z_0| < R$, but not at z_0 itself. Laurent's Theorem tells us that $f(z)$ equals its Laurent series on said domain, so long as the curve C encircles z_0.

The *residue* of $f(z)$ at z_0 is the coefficient a_{-1}

$$\text{Res}_{z=z_0} f(z) = a_{-1} = \frac{1}{2\pi i} \oint_C f(z) dz$$

The following forms of singularities can be retrieved from the Laurent Series

Removable singularity: The Laurent series is a Taylor series, meaning there are no negative powers and the residue is naturally zero. The series $f(z)$ extends analytically to z_0.

Pole of order m: The highest negative power in the Laurent series is $(z - z_0)^{-m}$. A *simple pole* is a pole of order 1, *double pole* for order 2, etc...

Essential singularity: The Laurent series has infinitely many negative terms

Example 63: Determine the type of singularity of $z_0 = 0$ in $f(z) = \frac{\sin z}{z}$.

For this problems, we want to first expand out the Laurent Series and search for negative terms, if they exist.

$$f(z) = \frac{1}{z} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n} = 1 - \frac{1}{6} z^2 + ...$$

There are no negative terms, so $z_0 = 0$ is a *removable singularity.* Consequently, $\text{Res}_{z=0} = 0$.

Example 64: Determine the type of singularities of $f(z) = \frac{1}{z(z-i)^2}$.

To obtain a better understanding, we rewrite f using partial fractions:

$$f(z) = \frac{1}{z(z-i)^2} = \frac{A}{z} + \frac{B}{z-i} + \frac{C}{(z-i)^2}.$$

There is a simple pole at $z = 0$ with $\text{Res}_{z=0} = A$ and a double pole at $z = i$ with $\text{Res}_{z=i} = B$ (remember we only look at the coefficient for the $\frac{1}{z-i}$ term!).

Example 65: Identify the type of singularity for $f(z) = \frac{1-e^{2z}}{z^4}$ and compute the residue.

Once again, find the Laurent Series of $f(z) = 1 - e^{2z}$ and divide by z^4.

$$1 - e^{2z} = 1 - \sum_{n=0}^{\infty} \frac{(2z)^n}{n!} = \sum_{n=1}^{\infty} \frac{(2z)^n}{n!} \implies \frac{1}{z^4}(1 - e^{2z}) = \sum_{n=1}^{\infty} \frac{2^n z^{n-4}}{n!} = -\sum_{n=-3}^{\infty} \frac{2^{n+4} z^n}{(n+4)!}.$$

Since the negative coefficients go up to a_{-3}, we have a pole of order 3 and

$$\text{Res}_{z=0} f(z) = a_{-1} = \frac{2^3}{3!} = \frac{4}{3}.$$

Example 66: Compute the residue at $z = 0$ of $g(z) = z\cos\left(\frac{1}{z}\right)$.

The Laurent Series of g is

$$g(z) = z \sum_{n=0}^{\infty} \frac{(-1)^n}{z^{2n}(2n)!} \sum_{n=0}^{\infty} \frac{(-1)^n}{z^{2n-1}(2n)!} \qquad Res_{z=0} g(z) = a_{-1} = -\frac{1}{2}$$

IMPORTANT: Find the n that gives the z^{-1} term, which in this case is if $n = 1$.

Theorem 6.6: Cauchy's Residue Theorem

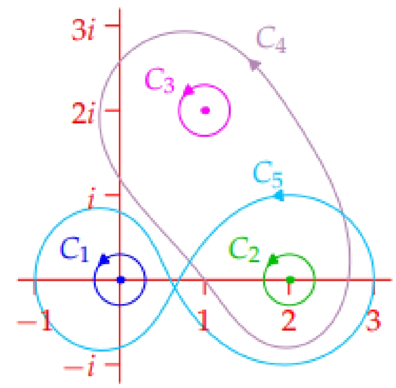
Let $f(z)$ be analytic on and inside a simple closed contour C, except at finitely many singularities $z_1, ..., z_n$. Then

$$\oint_C f(z)dz = 2\pi i \sum_{k=1}^{n} \text{Res}_{z=z_k} f(z).$$

If C is closed and orbits z_k counter-clockwise λ_k times, then

$$\int_C f(z)dz = 2\pi i \sum_{k=1}^{n} \lambda_k \text{Res}_{z=z_k} f(z).$$

Example 67: Look at the below diagram and compute $\oint_C \left(\frac{3}{z} + \frac{1}{z^2} + \frac{5i}{z-2} + \frac{1}{z-1-2i}\right) dz$.



$f(z)$ is clearly analytic except when $z = 0, 2,$ and $1 + 2i$. We compute $f(z)$ about each contour integral, or namely, $\left(\oint_{C_1} + \oint_{C_2} + \oint_{C_3} + \oint_{C_4} + \oint_{C_5}\right) f(z)dz$. We ultimately want to check whether the each contour contains the specified singularity. For example, $\frac{3}{z}$ has the simple pole $z = 0$, so

$$\oint_{C_1} f(z)dz = 3(2\pi i) = 6\pi i.$$

Recall that even though $\frac{1}{z^2}$ has a double pole at $z = 0$, its integral around C_1 is zero by the definition of residues. The other singularities lie outside C_1, so their integrals are also zero. Following similar logic for C_2 and C_3 give

$$\oint_{C_2} f(z)dz = 5i\oint_{C_2} \frac{dz}{z-2} = 5i(2\pi i) = -10\pi, \oint_{C_3} f(z)dz = \oint_{C_3} \frac{dz}{z-1-2i} = 2\pi i.$$

The integrals around C_4 and C_5 are more interesting. The analyticity of $f(z)$ on/in C_4 implies its analyticity over C_2 and C_3. Hence,

$$\oint_{C_4} = \oint_{C_2} dzf(z) + \oint_{C_3} f(z)dz = 2\pi(i-5)$$

For C_5, notice how C_5 encircles C_2 counter-clockwise but C_1 clockwise. Therefore,

$$\oint_{C_5} = \oint_{C_2} f(z)dz - \oint_{C_1} f(z)dz = 2\pi i - 10\pi - 6\pi i = -2\pi(5+3i)$$

Cauchy's Residue Theorem really tells us that the contours C_1, C_2, C_3 are encircled by other contours and we need only compute $\oint_{C_1}, \oint_{C_2}, \oint_{C_3}$ by a linear combination of how many times λ_k a singularity z_k is orbited in a counter-clockwise direction. So,

$$\int_C f(z)dz = \lambda_1 \oint_{C_1} f(z)dz + \lambda_2 \oint_{C_2} f(z)dz + \lambda_3 \oint_{C_3} f(z)dz = 6\pi i\lambda_1 - 10\pi\lambda_2 + 2\pi i\lambda_3.$$

Example 68: Use Cauchy's Residue Theorem to compute $\oint_C \frac{e^{-z}}{(z-1)^2}dz$ if C is the circle $|z| = 3$.
The (essential) singularity $z = 1$ lies within C, so we are free to apply Cauchy's Residue Theorem. We compute the residue by finding a_{-1}, for which we use Cauchy's Integral Formula

$$a_{-1} = \frac{1}{2\pi i}\oint_C \frac{e^{-z}}{(z-1)^2}dz = \frac{2\pi i}{2\pi i}f'(1) = -\frac{1}{e} \implies \oint_C \frac{e^{-z}}{(z-1)^2}dz = -\frac{2\pi i}{e}.$$

Example 69: Evaluate $\oint_C z^2 e^{\frac{1}{z}}dz$ where C is the circle C is the unit circle.

$$z^2 e^{\frac{1}{z}} = z^2\sum_{n=0}^{\infty}\frac{1}{z^n(n!)} = \sum_{n=0}^{\infty}\frac{1}{z^{n-2}(n!)} \implies \text{Res}_{z=0}f(z) = \frac{1}{3!} = \frac{1}{6} \implies \oint_C z^2 e^{\frac{1}{z}}dz = \frac{\pi i}{3}.$$

We use $n = 3$ to obtain the residue.

Example 70: Compute $\oint_C \frac{z+1}{z^2-2z}dz$ where C is the circle $|z| = 3$.

First note that $\frac{z+1}{z^2-2z} = \frac{z+1}{z(2-z)}$. With two different singularites, we must find the residues of $f(z)$ at $z = 0$ and $z = 2$.

We can find an expression for f using partial fractions:

$$f(z) = -\frac{1}{2z} + \frac{3}{2(z-2)} \implies \text{Res}_{z=0}f(z) = -\frac{1}{2}, \text{Res}_{z=2}f(z) = \frac{3}{2}$$

as they are just the z^{-1} coefficients. Hence, we have

$$\oint_C f(z)dz = 2\pi i\left(\text{Res}_{z=0}f(z) + \text{Res}_{z=2}f(z)\right) = 2\pi i.$$

Now, we want to look at the relationship between poles and residuals.

Theorem 6.7: Poles and Residuals, Part 1

Let $f(z)$ have a pole of order m at z_0. Then, $f(z) = (z-z_0)^{-m}\phi(z)$ where ϕ is analytic and nonzero. In such a case,

$$\text{Res}_{z=z_0}f(z) = \frac{\phi^{(m-1)}(z_0)}{(m-1)!}$$

In the case of a simple pole, Res $= \phi(z_0)$.

Let's look an at example in which we compute the integral of a function containing poles of different orders:

Example 71: Compute $\oint_C \left(\frac{3z(z-i)}{(z-2i)^2(z+1)}\right) dz$.

We investigate the residuals at zero points: The simple pole $z_1 = -1$ and $z_2 = 2i$.

At $z_1 = -1$, we need only compute $\phi(-1)$, where $\phi(z) = \frac{3z(z-i)}{(z-2i)^2}$. We have $\phi(-1) = \frac{3(i-1)}{(1+2i)^2} = \text{Res}_{z=-1} f(z)$.

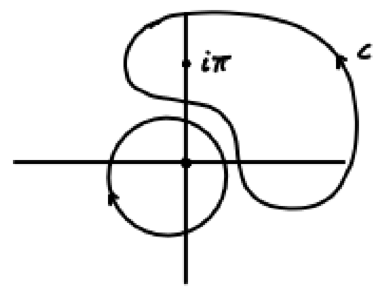
At the double pole $z_2 = 2i$, we need to compute $\phi'(z)$, where $\phi(z) = \frac{3z(z-i)}{z+1}$

$$\text{Res}_{z=2i} f(z) = \left.\frac{3[(2z-i)(z+i) - z(2-i)]}{(z+1)^2}\right|_{z=2i} = \frac{2}{(1+2i)^2}[3i(1+2i)+2]$$

Hence $\oint_C \left(\frac{3z(z-i)}{(z-2i)^2(z+1)}\right) dz = 2\pi i(\text{Res}_{z=-1} + \text{Res}_{z=2i})$.

Example 72: Compute $\oint_C \frac{e^{2z}}{z(z-i\pi)^2}$, where C is the following contour.

Notice that the circular contour is clockwise, so



$$\oint_C f(z) = 2\pi i\left(\text{Res}_{z=i\pi} f(z) - \text{Res}_{z=0} f(z)\right)$$

As for the simple pole $z_0 = 0$,

$$\text{Res}_{z=0} = \left.\frac{e^{2z}}{(z-i\pi)^2}\right|_{z=0} = -\frac{1}{\pi^2}$$

At the double pole $z = i\pi$,

$$\text{Res}_{z=i\pi} = \left.\frac{d}{dx}\frac{e^{2z}}{z}\right|_{z=i\pi} = \left.\frac{2ze^{2z} - e^{2z}}{z^2}\right|_{z=i\pi} = \frac{1-2\pi i}{\pi^2}.$$

By the residual definition we used above,

$$\oint_C f(z) = 2\pi i\left(\text{Res}_{z=i\pi} f(z) - \text{Res}_{z=0} f(z)\right) = \frac{2i}{\pi}(2-2\pi i) = 4\left(1+\frac{i}{\pi}\right).$$

Example 73: Let $P(z)$ and $Q(z)$ be polynomials and assume C is a simple closed contour such that all zeros of $Q(z)$ lie interior to C. If $\deg Q \geq 2 + \deg P$, prove that $\int_C \frac{P(z)}{Q(z)} dz = 0$.

Let $m = \deg P$ and $n = \deg Q$ such that $\deg Q \geq 2 + m$ and thus write $P(z) = \sum a_j z^j, Q(z) = \sum b_k z^k$. Then, by construction,

$$\frac{1}{2\pi i} \oint_C \frac{P(z)}{Q(z)} dz = \sum_{k=1}^{n} \operatorname*{Res}_{z=z_k} \frac{P(z)}{Q(z)} = \operatorname*{Res}_{z=0} \frac{P(z^{-1})}{z^2 Q(z^{-1})}$$

$$= \operatorname*{Res}_{z=z_0} \frac{a_m z^{-m} + \cdots + a_1 z^{-1} + a_0}{z^2 b_n z^{-n} + \cdots + b_1 z^{-1} + b_0} = \operatorname*{Res}_{z=z_0} \frac{z^n (a_m + a_{m-1} z + \cdots + a_0 z^m)}{z^{m+2} (b_n + b_{n-1} z + \cdots b_0 z^n)}$$

$$= \operatorname*{Res}_{z=0} \left(z^{n-m-2} f(z) \right)$$

whence $f(z)$ is analytic at zero. If $n \geq m + 2$, then $z^{n-m-2} f(z)$ is analytic at zero $\implies \operatorname{Res}_{z=z_0} f(z) = 0 \implies \frac{P(z)}{Q(z)} dz = 0$.

The following theorem helps us generalize simple poles to functions whose denominator is not a polynomial:

Theorem 6.8: Poles and Residuals, Part 2

Suppose $q(z)$ has a simple zero at z_0. Then, for $p(z_0) \neq 0$,

$$\operatorname{Res}_{z=z_0} \frac{p(z)}{q(z)} = \frac{p(z_0)}{q'(z_0)}.$$

Example 74: Find the residuals of $f(z) = \frac{z^2+4}{\sin z}$.

$f(z)$ has simple poles when $z = k\pi$ for $k \in \mathbb{Z}$. So,

$$\operatorname{Res}_{z=k\pi} f(z) = \frac{(k\pi)^2 + 4}{\cos(k\pi)} = (-1)^k (4 + k^2 \pi^2)$$

Theorem 6.9: Cauchy's Argument Principle

Suppose $f(z)$ is analytic except at poles, on and inside a simple close curve C, and that $f(z)$ has no poles on zeros on C. Then,

$$\frac{1}{2\pi i} \oint_C \frac{f'(z)}{f(z)} dz = Z - P$$

where Z is the number of zeros of f inside C and P is the number of poles of f inside C.

Example 75: Take $f(z) = \frac{(z-i)^2 \sin z}{(z-5)^4}$ where C is a circle $|z| = 6$. f has three zeros: two at $z = i$, and one at $z = 0$. f has a pole of order 4 at $z = 5$. So, by Cauchy's Argument Principle,

$$\frac{1}{2\pi i} \oint_C \frac{f'(z)}{f(z)} dz = 3 - 4 = -1.$$

This can be verified by integrating $\frac{f'(z)}{f(z)}$, namely through logarithmic differentiation.

We leave the following theorem as general results of the different types of singularities:

Theorem 6.10: Removable/Essential Singularities and Poles

Suppose $f(z)$ has an isolated singularity at z_0. The following are equivalent:

(1) The singularity is *removable.*

(2) $\lim\limits_{z \to z_0} f(z)$ exists and is finite.

(3) There exists a punctured disk $0 < |z - z_0| < \delta$ on which $f(z)$ is bounded.

Suppose z_0 is essential and that $w \in \mathbb{C} \cup \{\infty\}$ is given. Then, there exists a sequence (z_n) converging for which $\lim_{n \to \infty} f(z_n) = w$ (Casorati-Weierstrass).

z_0 is a *pole* if and only if $\lim_{z \to z_0} f(z) = \infty$.

Most of these results are (hopefully!) familiar from real analysis, but translated to the world of complex numbers.

7 Improper Integrals

Improper integrals are useful in embellishing a lot of the concepts introduced from the past six sections. We can use the tools we've learned about complex analysis to actually help compute improper integrals with real variables!

Definition 7.1: Cauchy's Principal Value

Suppose $f : \mathbb{R} \to \mathbb{R}$ is integrable. Provided the limit exists, the *Cauchy Principal Value* (CPV) of the improper integral $\int_{-\infty}^{\infty} f(x)dx$ is the limit

$$\text{P.V.} \int_{-\infty}^{\infty} f(x)dx = \lim_{R\to\infty} \int_{-R}^{R} f(x)dx.$$

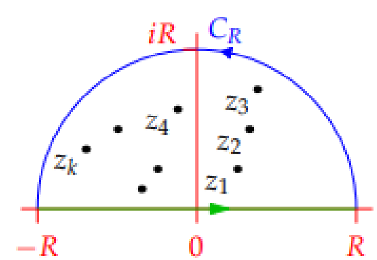
WARNING!: We know that if the standard integral converges, it equals its CPV. However, the converse is not guaranteed to be true!

Example 76: The function $f(x) = x^3$ is odd and we have that

$$\text{P.V} \int_{-\infty}^{\infty} x^3 dx = 0 \text{ but } \int_{0}^{\infty} x^3 dx \text{ diverges } \implies \int_{-\infty}^{\infty} x^3 dx \text{ diverges.}$$

As a result, we must be careful with odd functions!

We now consider a clever way of computing such improper integrals, mainly stemming from residue theory:



(1) Suppose $f(x)$ is the restriction to the real line of a complex function $f(z)$ which is analytic on the upper-half plane ($\text{Im}z \geq 0$) except at finitely many poles $z_1, ..., z_n$, none of which lie on the real axis.

(2) Choose $R > 0$ so that all poles z_k lie inside the curve formed by the real axis and the semi-circle C_R with radius R. By Cauchy's Residue Theorem,

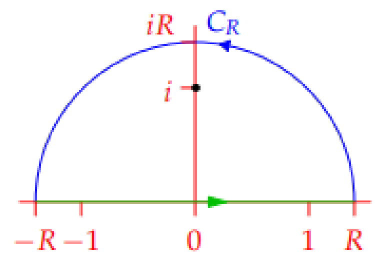
$$\int_{-R}^{R} f(x)dx + \int_{C_R} f(z)dz = 2\pi i \sum_{k=1}^{n} \operatorname*{Res}_{z=z_k} f(z).$$

(3) If $\lim_{R\to\infty} \int_{C_R} f(z)dz = 0$, then $\text{P.V.} \int_{-\infty}^{\infty} f(x)dx = 2\pi i \sum_{k=1}^{n} \operatorname*{Res}_{z=z_k} f(z)$.

We observe that knowledge of poles/residues, ML-inequality, and parametrization of curves are all imperative in the following problems.

Example 77: Evaluate $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{x^2+1}$ using Cauchy's Residue Theorem and Principal Value.

Translating to complex numbers, $\frac{1}{z^2+1}$ has simple poles $z = \pm i$. Now, we consider a semi-circle $C_R = |z| = R > 1$ on the upper-half plane. $f(z)$ is certainly analytic over C_R, and the poles of f are well-contained in C_R. We establish an upper bound for $\left|\oint_{C_R} f(z)dz\right|$ using the ML-inequality:



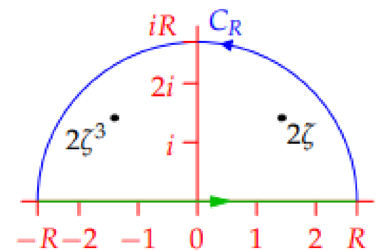
$$|z^2+1| \geq \big||z|^2 - 1\big| = R^2 - 1 \Longrightarrow \frac{1}{|z^2+1|} \leq \frac{1}{R^2-1} \Longrightarrow \left|\oint_{C_R} f(z)dz\right| \leq \frac{R\pi}{R^2-1} \xrightarrow[R\to\infty]{} 0.$$

By verifying the limit, we can now compute the Cauchy Principal Value

$$\text{P.V.} \int_{-\infty}^{\infty} \frac{dx}{x^2+1} = 2\pi i \operatorname*{Res}_{z=i} f(z) = \frac{2\pi i}{2i} = \pi.$$

Example 78: Compute $\displaystyle\int_{-\infty}^{\infty} \frac{4(x^2-1)}{x^4+16} dz$

First, we note that the simple poles of $f(z)$ are $\pm 2\zeta, \pm 2\zeta^3$, where $\zeta = e^{\frac{\pi i}{4}}$. If we choose $C_R = |z| = R > 2$ on the upper-half plane, is it obvious to infer that $f(z)$ is analytic on C_R and $2\zeta, 2\zeta^3$ are both contained in C_R. By Theorem 6.8, let $p(z) = 4(z^2-1)$ and $q(z) = z^4 + 16$ such that



$$\operatorname*{Res}_{z=z_0} = \frac{z_0^2-1}{z_0^3}$$

Using the ML-inequality:

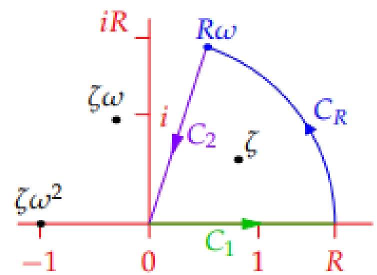
$$|z^4+16| \geq \big||z|^4 - 16\big| = R^4 - 16 \Longrightarrow \left|\oint_{C_R} f(z)dz\right| \leq \frac{4\pi R(R^2-1)}{R^4-16} \xrightarrow[R\to\infty]{} 0.$$

Hence, we can compute the CPV:

$$\text{P.V.} \int_{-\infty}^{\infty} f(x)dx = 2\pi i \left(\operatorname*{Res}_{z=2\zeta} f(z) + \operatorname*{Res}_{z=2\zeta^3} f(z)\right) = 2\pi i \left(\frac{4\zeta^2-1}{8\zeta^3} + \frac{4\zeta^6-1}{8\zeta^9}\right) = \frac{3\pi}{2\sqrt{2}}.$$

While this method works, we run into problems if the simple poles lie on the positive or negative real axis, as we will see with the next example.

Example 79: Compute $\displaystyle\int_0^\infty \frac{dx}{x^5+1}$.



There are five simple poles – $\zeta = e^{\frac{i\pi}{5}}, \zeta\omega^k$ where $\omega = e^{\frac{2\pi i}{5}}$ ($k = 1, 2, 3, 4$) – which are plainly the fifth roots of -1. However, note that $\zeta\omega^2 = -1$, so we can't let C_R be a semi-circle! Now, modify the semi-circle to a sector whose area is $\frac{1}{5}$ of a full circle. That way, we only deal with one pole. However, we must parametrize the contour into C_R, C_2, and C_1. Parametrize C_2 as $z(t) = t\omega$ from $0 \le t \le R$ and we find

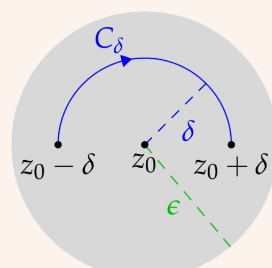
$$\int_{C_2} \frac{dz}{z^5+1} = \int_R^0 \frac{\omega}{t^5+1} dt = -\omega \int_0^R \frac{dt}{t^5+1} = -\omega \int_{C_1} \frac{dz}{z^5+1}$$

$$\implies (1-\omega)\int_0^R \frac{dx}{x^5+1} + \int_{C_R} \frac{dz}{z^5+1} = 2\pi i \operatorname*{Res}_{z=\zeta} \frac{1}{z^5+1} = \frac{2\pi i}{5\zeta^4} = \frac{2\pi i}{5\omega^2}.$$

For $|z| = R > 1$, $|z^5+1| \ge R^5 - 1 \implies \left|\int_{C_R} \frac{dz}{z^5+1}\right| \le \frac{2\pi R}{5(R^5-1)} \xrightarrow[R\to\infty]{} 0.$. Therefore,

$$\int_0^\infty \frac{dx}{x^5+1} = \frac{2\pi i}{5(\omega^2-\omega^3)} = \frac{2\pi i}{5\zeta\omega^2(\zeta^{-1}-\zeta)} = \frac{2\pi i}{5\left(2i\sin\frac{\pi}{5}\right)} = \frac{\pi}{5}\csc\left(\frac{\pi}{5}\right).$$

Lemma 7.2: Indented Paths



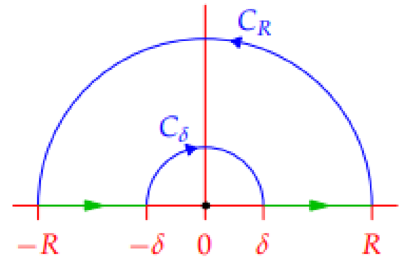
Let D be the disk $|z - z_0| \le \varepsilon$ and let $\delta < \epsilon$, and let C_δ be the clockwise semi-circle.

1. If $\phi(z)$ is analytic on D, then $\displaystyle\lim_{\delta\to 0}\int_{C_\delta} \phi(z)dz = 0$.

2. If $f(z)$ is analytic on $D\backslash\{z_0\}$ with a simple pole at z_0, then
$$\lim_{\delta\to 0}\int_{C_\delta} f(z)dz = -\pi i \operatorname*{Res}_{z=z_0} f(z).$$

More generally, if C_δ spans θ radians clockwise around z_0, then $\displaystyle\lim_{\delta\to 0}\int_{C_\delta} f(z)dz = -i\theta \operatorname*{Res}_{z=z_0} f(z)$.

Example 80: Compute $\displaystyle\int_0^\infty \frac{1+\sqrt{x}}{x^2+1}dx$ using the indented path $0 < \delta < 1 < R$, as pictured below. Assume $\sqrt{x}$ is the principal square root.



The simple poles are located at $z = \pm i$. Note that

$$\int_0^\infty \frac{1+\sqrt{x}}{x^2+1}dx = \int_0^\infty \frac{dx}{x^2+1} + \int_0^\infty \frac{\sqrt{x}}{x^2+1}dx$$

The first integral comes out to $\frac{\pi}{2}$ by Example 77. We will need to apply the indented path to compute the second integral. Apply the residue theorem to the simple pole $z = i$.

$$\operatorname*{Res}_{z=i} f(z) = \frac{e^{\frac{i\pi}{4}}}{2i} = \frac{1+i}{2i\sqrt{2}}$$

We have that

$$\left(\int_{C_\delta} + \int_{C_1} + \int_{C_2} + \int_{C_R}\right) f(z)dz = 2\pi i \operatorname*{Res}_{z=i} f(z) = \frac{1+i}{\sqrt{2}}.$$

We can control the size of $\displaystyle\int_{C_\delta}$ and $\displaystyle\int_{C_R}$ and show that their values converge to 0 by the ML-inequality

$$\left|\int_{C_R} f(z)dz\right| \leq \frac{\pi R\sqrt{R}}{R^2-1} \xrightarrow[R\to\infty]{} 0, \qquad \left|\int_{C_\delta} f(z)dz\right| \leq \frac{\pi\delta\sqrt{\delta}}{\delta^2-1} \xrightarrow[\delta\to 0^+]{} 0$$

We will need to establish a relationship between $\displaystyle\int_{C_1}$ and $\displaystyle\int_{C_2}$, where C_1 is the path from $-R \to -\delta$, and C_2 is the path from $\delta \to R$. This can be obtained through parameterization. If we choose $z(t) = -t = te^{i\pi}$, then $z'(t) = -1$ and $f(z(t)) = \frac{\sqrt{-t}}{(-t)^2+1}$. Then,

$$\int_{C_1} f(z)dz = -\int_{-C_1} f(z)dz = \int_\delta^R \frac{\sqrt{-t}}{(-t)^2+1}dt = \int_\delta^R \frac{i\sqrt{t}}{t^2+1}dt = i\int_\delta^R \frac{\sqrt{t}}{t^2+1}dt = i\int_{C_2} f(z)dz$$

Since $\int_{C_\delta}$ and $\int_{C_R} \to 0$, we finally have

$$(1+i)\int_0^\infty \frac{\sqrt{x}}{x^2+1}dx = \frac{\pi(1+i)}{\sqrt{2}} \Longrightarrow \int_0^\infty \frac{\sqrt{x}}{x^2+1}dx = \frac{\pi}{\sqrt{2}}.$$

Lastly,

$$\int_0^\infty \frac{1+\sqrt{x}}{x^2+1}dx = \pi\left(\frac{1}{2} + \frac{1}{\sqrt{2}}\right).$$

8 More Examples

We cover 20 more examples that span Sections 1 - 7.

Example 81: Find every root of $(-8-8i\sqrt{3})^{\frac{1}{4}}$ and exhibit them as vertices of a certain square, and point out which is the principal root.

Recall that $z = |z|e^{i\theta}$. Hence,

$$(-8-8i\sqrt{3})^{\frac{1}{4}} = \left(16e^{\frac{4\pi i}{3}}\right)^{\frac{1}{4}} = 2e^{\frac{\pi i}{3}} = 1+i\sqrt{3}$$

This is the principal root. To compute the other $k=3$ roots, we add $\frac{k\pi}{2}$ to $\frac{\pi}{3}$ for $k=1,2,3$. The remaining fourth roots are

$$z^{\frac{1}{4}} = 2e^{\frac{5\pi i}{6}} = -\sqrt{3}+i, 2e^{\frac{4\pi i}{3}} = -1-i\sqrt{3}, 2e^{\frac{11\pi}{6}} = \sqrt{3}+i.$$

As vertices of a square, they are $\pm(\sqrt{3}-i), \pm(1+i\sqrt{3})$.

Example 82: Prove that $\lim\limits_{z\to\infty} \dfrac{4z^2}{(z-1)^2} = 4$ using the $\varepsilon - \delta$ definition.

Proof: Let $t = \frac{1}{z}$, then $\lim\limits_{t\to 0} \dfrac{4}{t^2\left(\frac{1}{t^2}-\frac{2}{t}+1\right)} = \lim\limits_{t\to 0}\dfrac{4}{(t-1)^2}$. By the limit definition, let $\varepsilon > 0, \delta > 0$ be given such that

$$\left|\frac{4}{(t-1)^2}-4\right| = \left|\frac{2t-t^2}{(t-1)^2}\right| < \frac{|t||t-2|}{|t-1|} < \varepsilon \text{ for some } |t| < \delta.$$

Fix $\delta = \frac{1}{2}$. Then $-\frac{1}{2} < t < \frac{1}{2}$ and,

$$\frac{|t||t-2|}{|t-1|} = 5\delta < \varepsilon \implies \delta = \frac{\varepsilon}{5}.$$

Hence, choose δ such that

$$\delta = \min\left\{\frac{\varepsilon}{5}, \frac{1}{2}\right\}.$$

Example 83: Determine where $f'(z)$ exists and find its value when $f(z) = z\text{Im}z$.

Apply the Cauchy Riemann equations. Let $f(z) = f(x,y) = (x+iy)y = xy+iy^2$. $u_x = y = 2y = v_y$ when $y=0$. $u_y = x = 0 = -v_x$ when $x=0$. Hence the point $(x,y) = 0 \implies z = 0$ is the only point that satisfies the C-R equations. Because all partial derivatives are continuous, we say that $f(z)$ is differentiable at $z=0$ with

$$f'(0) = u_x + iv_x\big|_{(x,y)=(0,0)} = 0.$$

Example 84: Use polar coordinates to show that $f(z) = \frac{1}{z^4}$ is differentiable everywhere except for when $z \neq 0$ and then find $f'(z)$.

Verifying the Cauchy-Riemann equations in polar coordinates is a simple result of the multivariable chain rule:

Aside 8.1: Cauchy-Riemann Requirements in Polar Form

If we let $z = x + iy$, we have the criteria $u_x = v_y, u_y = -v_x$. In polar form, we let $z = re^{i\theta}$ and differentiate u and v with respect to r and θ. The multivariable chain rule gives us

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r}, \frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \theta}$$

If we assume that the partial derivatives of u and v with respect to x and y also satisfy the Cauchy-Riemann equations, then

$$ru_r = v_\theta, u_\theta = -rv_r.$$

For the points z_0 that satisfy the criteria above,

$$f'(z_0) = e^{-i\theta}(u_r + iv_r)\Big|_{(r_0,\theta_0)}.$$

Using the aside, we say $f(z) = \frac{1}{r^4}e^{-4i\theta} = \frac{\cos(4\theta)}{r^4} - \frac{\sin(4\theta)}{r^4}i$. Now we compute the partial derivatives:

$$ru_r = -\frac{4\cos(4\theta)}{r^4} = v_\theta, \qquad u_\theta = -\frac{4\sin(4\theta)}{r^4} = -rv_r$$

Hence the equations are satisfied everywhere except for when $r = 0$ because the partial derivatives are continuous for $r \neq 0$, as expected. So, we compute $f'(z_0)$ accordingly

$$f'(z_0) = e^{-i\theta_0}\left(-\frac{4\cos(4\theta_0)}{r_0^5} + \frac{4\sin(4\theta_0)}{r_0^5}\right) = e^{-i\theta_0}\left(-\frac{4}{r_0^5}e^{-4i\theta_0}\right) = -\frac{4}{r_0^5}e^{-5i\theta_0} = -\frac{4}{z_0^5}.$$

where $z_0 \in \mathbb{C}\backslash\{0\}$.

Example 85: Find the principal value of $\left[\frac{e}{2}(-1-i\sqrt{3})\right]^{3\pi i}$.

Let $f(z) = \left[\frac{e}{2}(-1-i\sqrt{3})\right]^{z}$, where

$$f(z) = \left(e \cdot e^{-\frac{2\pi i}{3}}\right)^2 = e^{z\text{Log}\left(e \cdot e^{-\frac{2\pi i}{3}}\right)} = e^{z\left(1-\frac{2\pi i}{3}\right)} \implies f(3\pi i) = e^{3\pi i\left(1-\frac{2\pi i}{3}\right)} = e^{3\pi i}e^{2\pi^2} = -e^{2\pi^2}.$$

Example 86: Compute $\sin^{-1}(-i)$.

Rewrite this as

$$\sin z = -i \iff \frac{e^{iz}-e^{-iz}}{2i} = -i \iff e^{iz} - e^{-iz} = 2.$$

Fix $u = e^{iz}$. Then we will obtain a quadratic in e^{iz}

$$u^2 - 2u - 1 = 0 \iff u = \frac{2 \pm \sqrt{8}}{2} = 1 \pm \sqrt{2} \iff e^{iz} = 1 \pm \sqrt{2} \iff z = -i\ln\left(1 \pm \sqrt{2}\right) + 2\pi n$$

where $n \in \mathbb{Z}$.

Example 87: By integrating the Taylor series for z^{-1} about $z_0 = 1$, prove that

$$\text{Log}z = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}(z-1)^n \qquad \text{whenever } |z-1| < 1.$$

We have that $f^{(n)}(1) = (-1)^n n!$, so the Taylor series for $\frac{1}{z}$ is

$$\frac{1}{z} = \sum_{n=0}^{\infty} \frac{(-1)^n n!(z-1)^n}{(n+1)!} \implies \int \frac{dz}{z} = \sum_{n=0}^{\infty} \int \frac{(-1)^n}{n}(z-1)^n dz = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)}(z-1)^{n+1}$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}(z-1)^{n+1} = \text{Log}z.$$

For Examples 88-90, let $f(z) = \frac{1-2i}{(z-1)(z-2i)} = \frac{1}{z-1} - \frac{1}{z-2i}$.

Example 88: Find a Laurent series centered at z_0 for $f(z)$ on $D_1 = \{z : 0 < |z| < 1\}$. Then, compute $\oint_C f(z)dz$ where C is a simple closed curve in the given domain encircling the origin.

We expand each series as a Maclaurin series

$$f(z) = -\frac{1}{1-z} + \frac{1}{2i-z} = -\frac{1}{1-z} + \frac{1}{2i\left(1-\frac{z}{2i}\right)} = \sum_{n=0}^{\infty}\left[(-1)^{n+1} + (2i)^{-(n+1)}\right]z^n.$$

$f(z)$ is analytic $\implies a_{-1} = 0 \implies \oint_C f(z)dz = 0$.

Example 89: Follow the same instructions as the previous example, except on the region $D_2 = \{1 < |z| < 2\}$

The second series from the previous example remains the same. However, we must manipulate the first series accordingly:

$$\frac{1}{z-1} + \frac{1}{2i-z} = \frac{1}{z}\left(\frac{1}{1-\frac{1}{z}}\right) + \frac{1}{2i-z} = \sum_{n=1}^{\infty}\frac{1}{z^n} + \sum_{n=1}^{\infty}(2i)^{-n}z^{n+1}$$

By residue theory, $\oint_C f(z)dz = 2\pi i a_{-1} = 2\pi i$.

Example 90: Follow the same instructions as the past two examples, except on the region $D_3 = \{z : |z| > 2\}$

The first series from the previous example remains the same. We manipulate the second series the same way we did in the previous example.

$$\frac{1}{z-1} - \frac{1}{z-2i} = \frac{1}{z-1} - \frac{1}{z}\left(\frac{1}{1-\frac{2i}{z}}\right) = \sum_{n=1}^{\infty}\frac{1}{z^n} - \sum_{n=1}^{\infty}\frac{(2i)^{n-1}}{z^n} = \sum_{n=1}^{\infty}\frac{1-(2i)^{n-1}}{z^n}.$$

The z^{-1} term vanishes, and so $\oint_C f(z)dz = 0$.

Compute the residue or residues for Examples 91 - 92.

Example 91: $f(z) = \frac{\sin(2z)}{(z-i)^3}$.

Let $\phi(z) = \sin(2z)$. Then,

$$\operatorname*{Res}_{z=i} f(z) = \phi''(i) = -4\sin(2i) = -4\left(\frac{e^{i^2}-e^{-i^2}}{2i}\right) = 2i(e^{-1}-e).$$

Example 92: $f(z) = \frac{z}{(z-2)^2(z+i)}$

We compute two residues: one at the simple pole $z = -i$ and the double pole $z = 2$.

$$\operatorname*{Res}_{z=-i} f(z) = -\frac{i}{(-i-2)^2} = -\frac{i}{3-4i} = -\frac{3i+4}{25}.$$

$$\operatorname*{Res}_{z=2} = f'(2) = \left.\frac{(z+i)-z}{(z+i)^2}\right|_{z=2} = \frac{i}{(2+i)^2} = \frac{4+3i}{25}.$$

For the remaining exercises, compute each integral using any of the following methods: Parameterization, Cauchy-Goursat, Cauchy's Integral Formula, Laurent Series, Residue Theorem.

Example 93: Compute $\int_C (\overline{z} + z^2)dz$ where C is the line from $1 + 3i$ to the origin.

Parameterization is not required for z^2, so we can directly compute the integral

$$\int_{1+3i}^{0} z^2 dz = -\frac{1}{3}(1+3i)^3 = -\frac{1}{3}(1+9i-27-27i) = \frac{26}{3} + 6i.$$

For the complex function $\overline{z}$ we must parameterize the line. Let $z(t) = (t-1)(1+3i)$. Then, $z'(t) = (1+3i)t$ and $f(z(t)) = t(1-3i) - (1-3i)$ and we hence compute the integral

$$\int_C \overline{z} dz = \int_0^1 (1+3i)(t(1-3i)-(1-3i))dt = \int_0^1 (10t-10)dt = -5.$$

Combining both integrals gives

$$\int_C (\overline{z} + z^2)dz = \frac{11}{3} + 6i.$$

Example 94: Let C be the square with vertices $\pm 2(1+i), \pm 2(-1+i)$. Evaluate $\oint_C \frac{z^2+i}{(z-3)^4} dz$.

The singularity $z = 3$ lies outside C. By Cauchy-Goursat, $\oint_C \frac{z^2+i}{(z-3)^4} dz = 0$.

Example 95: Let C_1 be the unit circle oriented counterclockwise and C_2 be a circle of radius 3 centered at the origin also oriented counterclockwise. Show $\oint_{C_1} \frac{\sqrt{z}}{z-\frac{1}{4}} dz = \oint_{C_2} \frac{\sqrt{z}}{z-\frac{1}{4}} dz$ and then compute the integral.

C_1 and C_2 are both positively oriented and non-intersecting, and the singularity $z = \frac{1}{2}$ lies interior to both contours. By Theorem 4.7, the two integrals are equal. Now, we use Cauchy's Integral Formula

$$\oint_{C_1} \frac{\sqrt{z}}{z-\frac{1}{4}} dz = \oint_{C_2} \frac{\sqrt{z}}{z-\frac{1}{4}} dz = 2\pi i \sqrt{\frac{1}{4}} = \pi i.$$

Example 96: Compute $\oint_C \frac{z^3+\sin(iz)}{(z-\pi)^4}dz$ where C is the circle $|z|=4$.

Use Cauchy's Integral Formula. Let $f(z)=z^3+\sin(iz)$. Then,

$$\oint_C \frac{z^3+\cos(iz)}{(z-\pi)^4}dz = \frac{2\pi i}{3!}f'''(\pi) = 2\pi i\left(1-\frac{\cos(i\pi)}{6}\right) = \pi i\left(2-\frac{1}{6}(e^{\pi}-e^{-\pi})\right)$$

Example 97: Compute $\oint_C \frac{e^{z^2}}{z^7}dz$ where C is the unit circle oriented counterclockwise.

The easiest way to compute this is through deriving the Laurent Series and find $2\pi i a_{-1}$.

$$\frac{e^{z^2}}{z^7} = \sum_{n=0}^{\infty}\frac{z^{2n}}{z^7 n!} = \sum_{n=0}^{\infty}\frac{z^{2n-7}}{n!} \qquad a_{-1}=\frac{1}{24} \Longrightarrow \oint_C \frac{e^{z^2}}{z^7}dz = \frac{i\pi}{12}.$$

Example 98: Compute $\oint_C \frac{3z^3+2}{(z-1)(z^2+9)}dz$ where C is the circle $|z|=4$.

We need to compute three residues, at $z=1, z=\pm 3i$

$$\operatorname*{Res}_{z=1} f(z) = \left.\frac{3z^2+2}{z^2+9}\right|_{z=1} = \frac{1}{2}.$$

$$\operatorname*{Res}_{z=3i} f(z) = \left.\frac{3z^2+2}{z-1}\right|_{z=3i} = \frac{3(3i)^3+2}{(3i-1)(6i)} = \frac{-81i+2}{-18-6i} = \frac{1}{60}(243i-6+2i+81) = \frac{15+49i}{12}$$

Similar computation yields

$$\operatorname*{Res}_{z=-3i} f(z) = \frac{15-49i}{12}.$$

Therefore, summing up the residues and multiplying by $2\pi i$:

$$\oint_C \frac{3z^3+2}{(z-1)(z^2+9)}dz = 2\pi i\left(\operatorname*{Res}_{z=1} f(z) + \operatorname*{Res}_{z=3i} f(z) + \operatorname*{Res}_{z=-3i} f(z)\right) = 6\pi i.$$

Example 99: Suppose that C is the rectangle whose sides are the lines $x=\pm 2, y=0$, and $y=1$. Compute $\oint_C \frac{dz}{(z^2-1)^2+3}$.

Solve the quartic function in the denominator:

$$z^4-2z+4 \xrightarrow[u=z^2]{} u^2-2u+4=0 \Longleftrightarrow u=1\pm i\sqrt{3}=2e^{\pm\frac{i\pi}{3}} \Longleftrightarrow z=\pm\sqrt{2}e^{\pm\frac{i\pi}{6}}.$$

The roots $w=\sqrt{2}e^{\frac{\pi i}{6}}, -\overline{w}=-\sqrt{2}e^{\frac{-\pi i}{6}}$ lie in C, so we compute the residues evaluated at those points. Since these are simple poles, we can simply evaluate at the derivative of the denominator:

$$\operatorname*{Res}_{z=w} = \frac{1}{4w(w^2-1)} = \frac{1}{4wi\sqrt{3}} \qquad \operatorname*{Res}_{z=-\overline{w}} = \frac{1}{4\overline{w}\sqrt{3}i}$$

Here we used the fact that $z^2 - 1 - i\sqrt{3} = 0 \Longrightarrow z^2 - 1 = i\sqrt{3}$ as provided from the function. Therefore,

$$\oint_C f(z)dz = \frac{2\pi i}{4i\sqrt{3}}\left(\frac{1}{w} + \frac{1}{\overline{w}}\right) = \frac{\pi}{2\sqrt{6}}\left(e^{-\frac{\pi i}{6}} + e^{\frac{\pi i}{6}}\right) = \frac{\pi}{\sqrt{6}}\cos\left(\frac{\pi}{6}\right) = \frac{\pi\sqrt{2}}{4}.$$

Example 100: Compute $\int_0^\infty \frac{x^2}{(x^2+9)(x^2+4)^2}dx$.

Let C be a semi-circle on the upper-half plane such that $C = |z| = R > 3, \text{Im}z \geq 0$. Then, for $R > 3$,

$$\left|\int_{C_R} f(z)dz\right| \leq \frac{\pi R \cdot R^2}{(R^2-9)(R^2-4)} \xrightarrow[R\to\infty]{} 0.$$

Therefore, we conclude that

$$\int_0^\infty f(x)dx = 2\pi i\left(\operatorname*{Res}_{z=3i} f(z) + \operatorname*{Res}_{z=2i} f(z)\right).$$

$f(z)$ has a simple pole at $z = 3i$. So,

$$\operatorname*{Res}_{z=3i} f(z) = \left.\frac{z^2}{(z+3i)(z^2+4)^2}\right|_{z=3i} = \frac{-9}{(6i)(-9+4)^2} = \frac{3i}{50}$$

We use logarithmic differentiation to compute the residue at the double pole $z = 2i$:

$$\operatorname*{Res}_{z=2i} f(z)dz = \left.\frac{d}{dz}\frac{z^2}{(z^2+9)(z+2i)^2}\right|_{z=2i}$$

Let $w = f(z)$. Then,

$$\ln w = 2\ln z - \ln\left(z^2+9\right) - 2\ln(z+2i) \Longrightarrow \frac{dw}{dz}\frac{1}{w} = \frac{2}{z} - \frac{2z}{z^2+9} - \frac{2}{z+2i}$$

$$\Longrightarrow \frac{dw}{dz} = w\left(\frac{2}{z} - \frac{2z}{z^2+9} - \frac{2}{z+2i}\right) = \left[\frac{z^2}{(z^2+9)(z+2i)^2}\right]\left(\frac{2}{z} - \frac{2z}{z^2+9} - \frac{2}{z+2i}\right)$$

Evaluated at $z = 2i$ gives $-\frac{13i}{200}$. Therefore, the improper integral

$$\int_0^\infty \frac{x^2}{(x^2+9)(x^2+4)^2}dx = 2\pi i\left(\frac{3i}{50} - \frac{13i}{200}\right) = \frac{\pi}{200}.$$

9 References

The notes (and visuals!) were taken and *heavily* influenced by *Complex Variables and Applications: 8th edition* by Brown and Churchill as well as Neil Donaldson's notes which can be found at `https://www.math.uci.edu/~ndonalds/`!