Introduction to Algebraic Geometry

DRP Festival Spring '25

Arjun Suresh, Michael Zhang, Ryan Gomberg

University of California, Irvine

June 4, 2025

Overview

Algebraic Varieties

Definition: Affine Algebraic Varieties

An *affine algebraic variety* is the common zero set of a collection $\{F_i\}_{i \in I}$ of complex polynomials on complex n-space $\mathbb{C}^n$. We write

$$V = \mathbb{V}\left(\{F_i\}_{i \in I}\right) \subset \mathbb{C}^n$$

for this set of common zeros.

Algebraic varieties are irreducible if they are completely factorized, meaning they cannot be expressed as the union of smaller algebraic sets.

- For instance, the variety resulting from the hyperbola $\mathbb{V}(x^2 - y^2 - 1)$ cannot be written as a product of linear polynomials in $\mathbb{C}$, so we say it is an irreducible variety.

Moreover, the **affine space** $\mathbb{A}^n$ is where affine varieties live. For example, the zero set of $y - x^2$ is defined by a parabola $y = x^2$ in $\mathbb{A}^2$.

Algebraic Varieties

Examples

A classical example is the twisted cubic curve. Define a map $\phi : \mathbb{A}^1 \to \mathbb{A}^3$ such that $t \mapsto (t, t^2, t^3)$. The curve is thereby defined by the image

$$x = t, y = t^2, z = t^3 \quad \implies \quad (t, t^2, t^3)$$

ϕ is injective and covers the **entire** variety. That is to say, every point on the twisted cubic is uniquely defined by some t.

Additionally, we can eliminate the parameter through the relationship $y = x^2, z = x^3$. The resulting variety

$$\mathbb{V}(y - x^2, z - x^3) \subset \mathbb{A}^3$$

captures the algebraic structure of the twisted cubic. Not only does it give the geometry of the curve, but also the set of polynomials that vanish on the curve.

Zariski Topology

Before proceeding, we discuss general properties of affine spaces. If we consider an n-dimensional affine space $\mathbb{A}^n$, we can observe that

- The empty set and $\mathbb{A}^n$ are affine algebraic varieties.
- The empty set is the zero set of $\mathbb{C}[x_1, \ldots, x_n]$ and $\mathbb{A}^n$ is the zero set of the empty set.

Any arbitrary (finite or infinite) *intersection* of affine algebraic varieties is an affine variety.

- The intersection of zero sets of polynomial systems is the zero set of the sum of these systems: $\mathbb{V}(I_1) \cap \ldots \cap \mathbb{V}(I_k) = \mathbb{V}(I_1 + \ldots + I_k)$.

Any finite *union* of affine algebraic varieties is an affine algebraic variety.

- The pairwise union of two zero sets of polynomial systems is the zero set of the product of the systems: $\mathbb{V}(I) \cup \mathbb{V}(J) = \mathbb{V}(IJ)$.

Zariski Topology

The previous results show that the affine algebraic varieties in $\mathbb{A}^n$ satisfy the axioms for a topological space when taken as closed subsets.

The resulting topology is called the **Zariski Topology** on $\mathbb{A}^n$.

- This is the reason we distinguish affine spaces $\mathbb{A}^n$ from $\mathbb{C}^n$: $\mathbb{A}^n$ is $\mathbb{C}^n$ equipped with the Zariski topology.

Algebraic Foundations - Ideals

An **ideal** is a subring that is closed under multiplication by anything from the larger ring. We say it is generated by a set

$$(J) = \bigcap \{I \mid J \subset I, I \subset R \text{ an ideal}\}.$$

An ideal I is **radical** if I is equal to the radical of I, which is defined as follows:

$$\sqrt{I} := \{f \in R \mid f^n \in I \text{ for some n } > 0\}.$$

Theorem: Hilbert's Basis Theorem

If a ring R is Noetherian (its ideals are finitely generated), then the polynomial ring over R in one variable, $R[x]$, is also Noetherian.

- This theorem allows us to describe any affine algebraic variety as the common zero of *finitely* many polynomials, rather than needing to be arbitrarily many. It follows from the property of Noetherian rings.
- Since the complex numbers are Noetherian, we show via induction that $\mathbb{C}[x_1, \ldots, x_n]$ is also Noetherian!

Combining Hilbert's Basis Theorem with the fact $\mathbb{V}(\mathbb{I}(V)) = V$, we can describe **any** affine algebraic variety as the zero locus of finitely many polynomials.

Our next objective is to equate algebraic and geometric ideas that have been presented so far. The upcoming theorem will help us do that!

Theorem: Hilbert's Nullstellensatz

For any ideal $I \subset \mathbb{C}[x_1, \ldots, x_n]$,

$$\mathbb{I}(\mathbb{V}(I)) = \sqrt{I} \quad \text{or if } I \text{ is radical, } \mathbb{I}(\mathbb{V}(I)) = I$$

- This explicitly creates a one-to-one correspondence between affine algebraic varieties in $\mathbb{A}^n$ and the radical ideals of the polynomial ring of n variables.
- And, in fact, this correspondence is order reversing!
- Moreover, Hilbert's Nullstellensatz holds for any algebraically closed field.

Algebraic Foundations - Coordinate Rings

Definition: Coordinate Rings

Given an algebraic variety V and any complex polynomial in n variables, the restriction to V defines a function $V \mapsto \mathbb{C}$. The quotient ring

$$\mathbb{C}[x_1, \ldots, x_n]/\mathbb{I}(V)$$

is formerly known as a coordinate ring, denoted by $\mathbb{C}[V]$.

Additionally, we define morphisms (polynomial maps) between varieties and naturally induced maps of the coordinate rings. If $F : V \to W$ is a morphism of affine varieties, then we have

$$\mathbb{C}[W] \to \mathbb{C}[V]; \quad g \mapsto g \circ F,$$

called the **pullback** of F.

Algebraic Foundations - Equivalence between Algebra and Geometry

- Each variety V determines its coordinate ring, and any morphism of varieties determines its pullback. So, the geometry determines the algebra
- The converse is also true!

Theorem: Isomorphism between Varieties and Coordinate Rings

Every finitely generated reduced $\mathbb{C}$-algebra is isomorphic to the coordinate ring of some affine algebraic variety. That is to say, if $V \xrightarrow{F} W$ is a morphism of affine algebraic varieties, then its pullback is a homomorphism between the coordinate rings $\mathbb{C}[W] \xrightarrow{F^\#} \mathbb{C}[V]$.

- Furthermore, if we have some $\sigma : R \to S$ (both finitely generated $\mathbb{C}$-algebras) then we can actually find a morphism F between the corresponding varieites such that σ is F's pullback
- This theorem yields a very important result: two varieties are isomorphic **iff** their coordinate rings are isomorphic

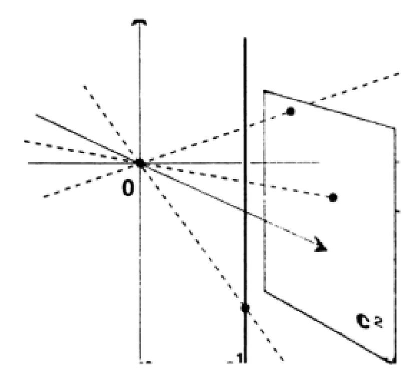
Projective Varieties

Definition: Projective Spaces

A complex projective space $\mathbb{P}^n$ is the set of all one-dimensional subspaces of $\mathbb{C}^{n+1}$, that is, it is the set of all lines in $\mathbb{C}^{n+1}$ passing through the origin. As a topological space, it is the quotient space

$$\mathbb{P}^n = \frac{\mathbb{C}^{n+1} \setminus \{0\}}{\sim}$$

where $\sim$ is the equivalence relation which identifies any two vectors that are nonzero multiples of each other.



Projective spaces aim to remove the limitation—of non-intersection, induced by the affine space—by allowing for the intersection of parallel lines, even if it is "at infinity."

Projective Varieties

- What this means is that we started with $\mathbb{C}^{n+1}$ equipped with the Euclidean topology, and then we "collapse" each equivalence class $\sim$ (a line passing through the origin) to a single point.
- The equivalence class in $\mathbb{P}^n$ represented by $(x_0, \ldots, x_n) \in \mathbb{C}^{n+1}$ is denoted by $[x_0 : x_1 : \cdots : x_n]$.
- The canonical map $\pi : \mathbb{C} \to \mathbb{P}$ which sends a point to its equivalence class is called a **quotient map**.
- We define the quotient topology on $\mathbb{P}$ by declaring that U is open in $\mathbb{P}^n$ **iff** $\pi^{-1}(U)$ is open in $\mathbb{C}^{n+1}$.

Projective Varieties

How do we define zero sets of polynomials in projective spaces?

- In general, the zero sets of a polynomial are not well-defined as subsets of complex projective space
  - For instance, $(x-1)(y-1) \in \mathbb{C}[x, y]$ has (1, 1) as a zero, but not (2, 2).
- For zero sets to be well-defined, we must restrict our attention to homogeneous polynomials, or polynomials whose terms are all of the same degree.
  - One can show this by scaling the inputs. If $F \in \mathbb{C}[x_1, \ldots, x_n]$ is homogeneous of degree d, then

$$F(\lambda x_0, \ldots, \lambda x_n) = \lambda^d F(x_0, \ldots, x_n).$$

  ...then if F vanishes at a nonzero point on a line, F vanishes at all other nonzero points.

We may thus define a projective algebraic variety in $\mathbb{P}^n$ to be the zero set of a collection of homogeneous polynomials in $\mathbb{C}[x_0, \ldots, x_n]$.

Quasi-Projective Varieties

Definition: Quasi-Projective Varieties

A Quasi-Projective Variety is an open subset of a projective variety in the Zariski topology. Or, equivalently, it can be viewed as a **locally closed subset** of projective space; that is, an intersection of an open subset and a closed subset of projective space $\mathbb{P}^n$.

With reference to the Zariski topology, it is defined on quasi-projective varieties by taking the closed sets to be the zero loci of sets of polynomials. Since quasi-projective varieties are open subsets of projective varieties, the Zariski topology on them is induced from the projective space.

Quasi-Projective Varieties

Generally, Quasi-Projective Spaces are the so-called “intermediate” to affine and projective varieties.

For example, we revisit the idea of whether spaces have “points at infinity.”

- As alluded to earlier, affine spaces have no points at infinity.
- Projective spaces include points at infinity by definition.
- Since quasi-projective spaces are open subspaces of projective spaces, they can include some, all, or none of the points at infinity, depending on the subset.

So, quasi-projective varieties interpolate well between affine and projective geometries.

Summary of Each Space

	Affine Spaces	Projective Spaces	Quasi-Projective Spaces
Defining Space	$\mathbb{A}^n$	$\mathbb{P}^n$	Open subset of $\mathbb{P}^n$
Coordinates	Cartesian	Homogeneous	Homogeneous on open subsets
Points at Infinity	Excluded	Included	Some may be excluded
Structure	Affine Variety	Projective Variety	Locally closed subset of $\mathbb{P}^n$
Ideal	Coordinate Ring $\mathbb{C}[x_1, \dots x_n]$	Homogeneous on $\mathbb{C}[x_0, \dots, x_n]$	
Topological Space	Zariski	Zariski	Zariski (induced)

The End