Math 140B Main Concepts: Ryan Gomberg

1 Continuity

Pointwise Continuity

We first start with the definition of real-valued function and domain/image. Given a function $f(x) : x \to \mathbb{R}$,

We call the domain of a real-valued function $\text{Dom}(f) = \mathbb{N}$. In general, the domain is $\text{Dom}(f) = (a,b), [a,b], (a,b], [a,b)$

The image, or range, is $\text{Im}(f) = f(I) = \{f(I) : x \in I\}$.
This definition helps us introduce the concept of pointwise continuity, as shown below.

Definition 1.1: Pointwise Continuity

We say f is **pointwise continuous** at $x = x_0 \in \text{Dom}(f)$ provided that for each $\epsilon > 0, \exists\delta > 0$ such that $|f(x) - f(x_0)| < \epsilon$ for all $x \in \text{Dom}(f)$ with $|x - x_0| < \delta$.

The idea is that δ depends on ϵ, so we want to find such δ as a function of ϵ.

Example 1: Show that $f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right), & x \neq 0 \\ 0, & x = 0 \end{cases}$ is pointwise continuous at $x_0 = 0$.

The first thing we notice is that $\left|\sin\left(\frac{1}{x}\right)\right| \leq 1 \ \forall x$, so we get

$$|f(x) - f(0)| = \left|x^2 \sin\left(\frac{1}{x}\right) - 0\right| \leq x^2$$

In addition we have that $|x - x_0| = |x| < \delta$. So, if we square $|x|$ then we get $|x|^2$ which we know is $< \epsilon$. Hence, let $\delta = \sqrt{\epsilon}$. We thus obtain

$$|f(x) - f(0)| \leq |x|^2 < \delta^2 = \epsilon.$$

This completes the proof.

Example 2: Show that $f(x) = x^2$ is pointwise continuous at $x_0 = a$ for $a \in \mathbb{R}$.

Let $\delta > 0, \epsilon > 0$ be given. We want to show that $|x - a| < \delta \implies |x^2 - a^2| < \epsilon$. To achieve this, we rewrite $|x^2 - a^2| = |x - a||x + a|$. Without loss of generality, let $|x - a| < 1$. This implies $|x + a| = |x - a + 2a| < 1 + 2|a|$. So, we have that

$$|x - a||x + a| < |x - a|(1 + 2|a|) = \delta \cdot (1 + 2|a|) = \epsilon.$$

We have two choices for $\delta : 1$ or $\frac{\epsilon}{1+2|a|}$. We want to choose whichever is smaller. So

$$\delta = \min\left\{1, \frac{\epsilon}{1+2|a|}\right\}$$

Theorem 1.2: Sequential Definition of Continuity

We say f is **pointwise continuous** at $x = x_0 \in \text{Dom}(f)$ if and only if for each sequence $\{x_n\} \subset \text{Dom}(f)$ converging to x_0, $\lim f(x_n) = f(x_0)$.

Example 3: Show that $f(x) = \begin{cases} \frac{1}{x}\sin\left(\frac{1}{x^2}\right), & x \neq 0 \\ 0, & x = 0 \end{cases}$ is discontinuous at $x = 0$.
Idea: We want to find a sequence x_n such that $x_n \to 0$ but $f(x_n)$ diverges. For simplicity, let us find such x_n for which $\sin\left(\frac{1}{x_n^2}\right) = 1$. For $x_n = \frac{1}{2\pi n + \frac{\pi}{2}}$, we have that $x_n \to 0$ but $\lim \frac{1}{x_n} \to \infty$. Therefore we have a discontinuity.

Theorem 1.3: Operations on Continuous Functions

Let f, g be continuous at x_0 and $c \in \mathbb{R}$ be some constant. We have that
(1) $|f|$ is continuous at x_0.
(2) cf is continuous at x_0.
(3) $f + g$ is continuous at x_0.
(4) $\frac{f}{g}$ is continuous at x_0 provided that $g(x_0) \neq 0$.
(5) $f(g(x)), g(f(x))$ are both continuous at x_0.

Properties of Continuous Functions
We look at two main properties of continuous functions

Theorem 1.4: Extreme Value Theorem

Let f be a continuous function on $[a, b]$. We have that f satisfies two properties.
(1) f is bounded.
(2) f attains its maximum/minimum on $[a, b]$ i.e. $\exists x_0, y_0 \in [a, b]$ such that $f(x_0) \leq f(x) \leq f(y_0)$.

The proof for this theorem stems from the Bolzano-Weierstrauss theorem and the properties of bounded sequences. For oscillating functions like $\sin x, \cos x$, we often have to show that we can find unique, converging subsequences because the main sequences diverge by oscillation.

Example 4: Show that $\sin(x)$ attains a maximum/minimum on $[0, 4\pi]$.

We obviously know that $\sin(x)$ is continuous. Let x_n be a sequence such that

$$x_n = \begin{cases} \frac{\pi}{2} + \frac{1}{n}, & n = 2k - 1 \\ \frac{5\pi}{2} + \frac{1}{n}, & n = 2k \end{cases}$$

Therefore we have that $x_n \to \frac{\pi}{2}, \frac{5\pi}{2}$ as $n \to \infty$ and $\lim f(x_n) = 1$. We have that $\sin(x) \leq 1$ for all $x \in [0, 4\pi]$. A similar case for the minimum can be applied for $x = \frac{3\pi}{2}, \frac{7\pi}{2}$.

Theorem 1.5: Intermediate Value Theorem (IVT)

Suppose f is continuous on an interval (a, b), then there exists a $c \in (a, b)$ such that for $f(a) < y < f(b)$, $f(c) = y$.

An incredibly useful result from the IVT is that we can use this to show the existence of roots of continuous functions. In fact, we can translate a lot of problems into root problems. Let the following example demonstrate this.

Example 5: Let f, g be two continuous functions on $[a, b]$ such that $f(a) \geq g(a)$ and $f(b) \leq g(b)$. Prove $f(x_0) = g(x_0)$ for at least one $x_0 \in [a, b]$.

Idea: Let us define a new function $h(x) = f(x) - g(x)$. At $h(a), f(a) \geq g(a)$ implies $h(a) > 0$, and $h(b) < 0$ by the same logic. Therefore, the IVT guarantees that $\exists x_0 \in [a, b]$ such that $h(x_0) = 0$. This implies $f(x_0) - g(x_0) = 0$ and $f(x_0) = g(x_0)$ for at least x_0, as desired.

Example 6: Suppose f is continuous on $[0, 2]$ and $f(0) = f(2)$. Prove there exist $x, y \in [0, 2]$ such that $|y - x| = 1$ and $f(x) = f(y)$.
Let $g(x) = f(x + 1) - f(x)$. We know g is continuous because $f(x + 1)$ is continuous on $[-1, 1]$ and $f(x)$ is continuous on $[0, 2]$. Computing $g(0)$ and $g(1)$ yields $g(0) = f(1) - f(0) > 0$ and $g(1) = f(2) - f(1) = f(0) - f(1) < 0$. Therefore we have $g(0) = -g(1)$. If $g(0) = 0$ then we are done. Otherwise, we see that $g(0)$ and $g(1)$ are of opposite signs. Therefore by IVT, there exists a $c \in (0, 1)$ such that $g(c) = 0$. Then $f(c + 1) - f(c) = 0 \implies f(c + 1) = f(c)$. If we let $y = c + 1$ and $x = c$ then the condition is satisfied.

Definition 1.6: Contractive Map

Let X be a set in $\mathbb{R}$. $f : X \to X$ is said to be a contractive map if there exists a $c \in (0, 1)$ such that $|f(x) - f(y)| < c|x - y|$

Definition 1.7: Banach Fixed Point Theorem

Let X be a closed set in $\mathbb{R}$ and $f : X \to X$ be a contractive map. Then, f has a unique fixed point in X.

Example 7: Determine whether $f(x) = \ln(1+x) : (0,\infty) \to (0,\infty)$ is a contractive map. We apply the definition of a contractive map to f. For $x, y \in (0,\infty)$, we have

$$|\ln(1+x) - \ln(1-y)| = \left|\ln\left(\frac{1+x}{1-y}\right)\right|$$

By the Mean Value Thoerem for derivatives, for some $z \in (x,y)$, there exists a c such that

$$|f(x) - f(y)| = \frac{1}{z+1}|x-y|$$

However, if we let y approach 0^+, then $\frac{1}{z+1} \to 1$, but we require it to be strictly less than 1. Therefore, we reach a contradiction.

Uniform Continuity

Definition 1.8: Uniform Continuity

Let a function f and interval I be given. We say f is **uniformly continuous** in I if for each $\epsilon > 0, \exists \delta > 0$ such that $|f(x) - f(y)| < \epsilon$ for all $x, y \in I$ with $|x - y| < \delta$. *Remark*: Recall that for pointwise continuity, δ is dependent on the point x_0 and ϵ. For uniform continuity, δ is not dependent on x, y, only ϵ!

Example 8: Show $f(x) = \frac{1}{x^2}$ is uniformly continuous on $[1,\infty)$.

Let $\epsilon > 0, \delta > 0$ be given. We want to find δ such that

$$\left|\frac{1}{x^2} - \frac{1}{y^2}\right| < \epsilon$$

We have that

$$\left|\frac{1}{x^2} - \frac{1}{y^2}\right| = \left|\frac{y^2 - x^2}{x^2y^2}\right| = \left|\frac{x+y}{x^2y^2}\right||x-y| = \left|\frac{1}{x^2y} + \frac{1}{xy^2}\right||x-y|$$

And so

$$\left|\frac{1}{x^2y} + \frac{1}{xy^2}\right||x-y| \leq 2|x-y| < \epsilon = 2\delta$$

Setting $\delta = \frac{\epsilon}{2}$ completes the proof.

Example 9: Show that $f(x) = \sin\left(\frac{2}{x}\right)$ is not uniformly continuous on $(0, 1)$.
Let $\epsilon = 2$. Then, we aim to find a $\delta > 0$ such that $|f(x) - f(y)| = \left|\sin\left(\frac{2}{x}\right) - \sin\left(\frac{2}{y}\right)\right| \geq 2$.
For $x = \frac{4}{\pi + 4\pi k}, y = \frac{4}{3\pi + 4\pi k}$, $k \in \mathbb{Z}$, we have $\left|\sin\left(\frac{2}{x}\right) - \sin\left(\frac{2}{y}\right)\right| = |1 - (-1)| = 2 \geq \epsilon$ where $|x - y| < \delta$ given $0 < x, y < \delta$. Therefore, $f(x)$ is not uniformly continuous on $(0, 1)$.

Theorem 1.9: Continuity implies Uniform Continuity on a Closed Interval

If f is continuous on $[a, b]$, then f is uniformly continuous on $[a, b]$.

Note that the converse is generally not true. Uniform continuity is a tighter case than continuity.

Example 10: Show that $f(x) = x^3$ is uniformly continuous on $[0, 1]$, but not uniformly continuous on $\mathbb{R}$.

(1) As for the first case, we simply apply the above Theorem. $f(x)$ is clearly continuous on $[0, 1]$, so its uniform continuity comes for free.

(2) The second case isn't as obvious. We need to use the sequential definition of continuity, so let $(x_n), (y_n)$ be two sequences such that $(x_n), (y_n) \to 0$. This time, we want to show $|f(x_n) - f(y_n)| \geq \epsilon$ for some $\epsilon > 0$ Let $x_n = n + \frac{1}{n}, y_n = n$. So $|f(x_n - f(y_n)| = \left|\left(n + \frac{1}{n}\right) - n^3\right| = 3n$. We know that $3n \geq 3$ for all $n \in \mathbb{N}$, so choosing $\epsilon = 3$ completes the proof.

Theorem 1.10: Uniform Continuity and Cauchy Sequences

If f is uniformly continuous on I and $\{s_n\}$ is a sequence in I, then $f(\{s_n\})$ is Cauchy.

Example 11: Show that $f(x) = \frac{1}{x^2}$ is not uniformly continuous on $(0, 1)$.

Here we let $s_n = \frac{1}{n}$, and so $f(s_n) = n^2$. Because s_n is a convergent sequence, it is also Cauchy. However, $f(s_n)$ is not Cauchy, so clearly $f(x)$ is not uniformly continuous on $(0, 1)$.

Theorem 1.11: Uniform Continuity and Differentiation

A function $f(x)$ is uniformly continuous on I if $|f'(x)| \leq M$ for some $M < \infty$.

Example 12: Show $f(x) = \sin(2x + 1) + \sqrt[3]{x}$ is uniformly continuous on $\mathbb{R}$.
First, we can apply the fact that the sum of two uniformly continuous functions is also uniformly continuous. So let us show (1) $g(x) = \sin(2x + 1)$ and (2) $h(x) = \sqrt[3]{x}$ are uniformly continuous functions on $\mathbb{R}$

(1) Intuitively, $|g'(x)| = |2\cos(2x+1)| \leq 2$ for all $x \in \mathbb{R}$. Because the derivative is bounded, clearly $\sin(2x+1)$ is uniformly continuous on $\mathbb{R}$.

(2) We need to be more careful with $h(x)$. In particular, we can investigate different intervals. First, look at $[-1, 1]$. It is obvious that h is continuous on this interval, so h is also uniformly continuous. Everywhere else, we show that $h'(x)$ is bounded. We have $|h'(x)| = \left|\frac{1}{3x^{\frac{2}{3}}}\right| \leq \frac{1}{3}$ for all $x \in (-\infty, -1] \cup [1, \infty)$, so $h(x)$ is uniformly continuous everywhere.

Because g and h are both uniformly continuous on $\mathbb{R}$, we have shown $f(x)$ is also uniformly continuous on $\mathbb{R}$.

2 Limits of Functions

Definition 2.1: $\epsilon - \delta$ Definition of Limits

We say that f has limit L when $x \to a$ if for any $\epsilon > 0$, there is a $\delta > 0$ such that for all $x \in (a-\delta, a+\delta) \cap X, x \neq a$, we have

$$|f(x) - L| < \epsilon \Longrightarrow \lim_{x \to a} f(x) = L$$

Remark: X is a subset on $\mathbb{R}$.

The format for proving limits is practically the same for continuity. We aim to find a $\delta > 0$ that $|x-a| < \delta$. This is commonly done by manipulating the function or finding bounds, as we will see in the following examples.

Example 13: Prove that $\lim_{x \to 4} x^3 = 64$.
Let $\epsilon > 0$ be given. We want to find a $\delta > 0$ such that $|x^3 - 64| < \epsilon$ for such $|x-4| < \delta$. Here we know $|x^3 - 64| = |(x-4)(x^2+4x+16)|$. We know that x^3 is strictly increasing so $|(x-4)(x^2+4x+16)| < |(x-4)(5^2+4(5)+16)| = 61|(x-4)|$. Therefore, let $\delta = \min\left\{1, \frac{\epsilon}{61}\right\}$. If $0 < |x-4| < \delta$, then $|x^3 - 64| < \epsilon$.

Example 14: Prove that $\lim_{x \to 1} \frac{x-1}{\sqrt{x}-1} = 2$.
Let $\epsilon > 0$ be given. We want to find a $\delta > 0$ such that $\left|\frac{x-1}{\sqrt{x}-1} - 2\right| = \left|\frac{x-1-2\sqrt{x}+2}{\sqrt{x}-1}\right| = \left|\frac{x-2\sqrt{x}+1}{\sqrt{x}-1}\right| = \left|\frac{(\sqrt{x}-1)^2}{\sqrt{x}-1}\right| = |\sqrt{x}-1| < \epsilon$. For $|x-1| < \delta$, we know $|\sqrt{x}-1| < \sqrt{\delta}$. Therefore, let $\delta = \epsilon^2$. If $0 < |x-1| < \delta$, then $\left|\frac{x-1}{\sqrt{x}-1} - 2\right| < \epsilon$.

Example 15: Prove that $\lim_{x \to 2} e^x = e^2$
For any $\epsilon > 0$, we want to find a $\delta > 0$ such that if $0 < |x-2| < \delta$ then $|e^x - e^2| < \epsilon$. We have

$$|e^x - e^2| = e^2|e^{x-2} - 1| = e^2\left|\sum_{n=0}^{\infty} \frac{(x-2)^n}{n!} - 1\right| \leq e^2 \sum_{n=1}^{\infty} \frac{1}{n!}|x-2| \leq e^3|x-2| < \epsilon$$

Therefore setting $\delta = \min\{e^{-3}\epsilon, 1\}$ gives us $|e^x - e^2| < \epsilon$ if $|x-2| < \delta$.

3 Power Series

Definition 3.1: Power Series and Radius of Convergence

A power series about $x = 0$ is written as

$$\sum_{n=0}^{\infty} a_n x^n$$

The **radius of convergence** of a power series is the interval in which the series converges, denoted by R. Let $R = \frac{1}{L}$ where

$$L = \limsup_{n\to\infty} |a_n|^{\frac{1}{n}}$$

Remark: R and L can be similarly computed with the Ratio Test and yields the same answer.

Example 16: Determine where $\sum \frac{2^n}{n} x^{3n}$ converges.
First we compute L

$$L = \limsup_{n\to\infty} \sqrt[3n]{\frac{2^n}{n}}$$

Each term is getting smaller and smaller, so the largest term, when $n = 1$, is the lim sup.

$$L = \pm\sqrt[3]{2} \implies R = \pm\frac{1}{\sqrt[3]{2}}$$

Now we check to see whether the endpoints converge. For both $x = -\frac{1}{\sqrt[3]{2}}, \frac{1}{\sqrt[3]{2}}$, we get the same series:

$$\sum 2^n \left(\pm\frac{1}{\sqrt[3]{2}}\right)^{3n} = \sum (\pm 1)^{3n}$$

which diverges. Therefore,

$$\sum \frac{2^n}{n} x^{3n} \text{ converges for } x \in \left(-\frac{1}{\sqrt[3]{2}}, \frac{1}{\sqrt[3]{2}}\right).$$

Example 17: Determine where $\sum \frac{3^n}{n^2} x^{n!}$ converges.
This time, we will make use of the ratio test

$$L = \limsup_{n\to\infty} \left|\frac{3^{n+1}x^{(n+1)!}}{(n+1)^2}\right| \left|\frac{n^2}{3^n x^{n!}}\right| = \left|\frac{3x^{(n+1)!-n!}n^2}{(n+1)^2}\right|$$

For $|x| < 1$, the limit approaches zero. Otherwise, it diverges (or cannot be defined if $x = 1$). So, $R = 1$.

If we plug in $x = \pm 1$, 3^n grows faster than n^2 so it will diverge at both endpoints. Therefore the series converges for $x \in (-1, 1)$.
Example 18: Determine where $\sum 2^n x^{n^2}$ converges.

$$L = \limsup_{n\to\infty} \left| \frac{2^{n+1}x^{(n+1)^2}}{2^n x^{n^2}} \right| = \limsup_{n\to\infty} \left|2x^{2n+1}\right|$$

Applying the same argument from Example 17, $R = 1$. If we check the endpoints, the geometric series diverges. Therefore the series converges for $x \in (-1, 1)$.

Example 19: Find the largest set in $\mathbb{R}$ such that $\sum \frac{(-1)^n}{3^n} x^{5n+1}$ converges.

$$L = \limsup_{n\to\infty} \left| \frac{x^{5(n+1)+1}3^n}{x^{5n+1}3^{n+1}} \right| = \lim_{n\to\infty} \left| \frac{x^5}{3} \right| < 1 \iff |x| < \frac{1}{\sqrt[5]{3}}$$

Plugging in the endpoints yields two geometric series that both converge.

$$\text{Therefore, } \sum_{n=1}^{\infty} \frac{(-1)^n}{3^n} x^{5n+1} \text{ converges for } x \in \left[-\frac{1}{\sqrt[5]{3}}, \frac{1}{\sqrt[5]{3}} \right]$$

Definition 3.2: Explicit Formulas for Power Series

Assume the power series $f(x) = \sum a_n x^n$ converges on $(-R, R) \subseteq \mathbb{R}$. Then $f(x)$ is differentiable and integrable on $(-R, R)$

(1) $\frac{d}{dx}\left(\sum_{n=0}^{\infty} a_n x^n\right) = \sum_{n=1}^{\infty} n a_n x^{n-1}$ for $|x| < R$
(2) $\int_0^x \sum_{n=0}^{\infty} a_n x^n dx = \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}$ for $|x| < R$
(3) $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$
(4) $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$
(5) $\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$

Example 20: Find the explicit expression of $f(x) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n} x^n$ for $x \in (-1, 1)$.
First we find $f'(x)$

$$f'(x) = \sum_{n=1}^{\infty} -\frac{(-1)^n}{n} n x^{n-1} = -\sum (-x)^{n-1} = -\sum (-x)^n = -\frac{1}{1+x}$$

Now we can integrate

$$\int_0^x f'(t)dt = -[\ln(1+t)]_0^x = -\ln(1+x) = f(x)$$

Example 21: Find the explicit expression of $\sum_{n=1}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1}$ for $x \in (-1, 1)$.

$$f'(x) = \sum_{n=1}^{\infty} (-1)^n x^{2n} = \sum \left(-x^2\right)^n = \frac{1}{1+x^2}$$

Integrating $f'(x)$ yields

$$\int_0^x f'(t)dt = \int_0^x \frac{dt}{1+t^2} = \tan^{-1}(x) = f(x)$$

Example 22: Find the explicit expression of $\sum_{n=1}^{\infty} (-1)^n n x^n$ for $x \in (-1, 1)$.

$$\sum_{n=1}^{\infty} (-1)^n n x^n = -x \left(\sum (-x)^n\right)' = -x \left(\frac{1}{1+x}\right)' = \frac{x}{(1+x)^2} = f(x)$$

4 Sequences of Functions and Convergence

Definition 4.1: Pointwise and Uniform Convergence

Let $\{f_n(x)\}_{n=1}^{\infty}$ be a sequence of functions on X.

We say that $f_n \to f$ **pointwisely** on X if $\lim_{n\to\infty} f_n(x) = f(x)$ for every given $x \in X$.

We say that $f_n \to f$ **uniformly** if $\sup\{|f_n(x) - f(x)| : x \in X\} \to 0$ as $n \to \infty$.

Remark: If f_n does not converge pointwisely, then it will not converge uniformly. Think of uniform convergence as a stronger case compared to pointwise convergence.

Example 23: Let $f_n(x) = x^n$. Determine if $f_n(x)$ converges uniformly on $(0, 1)$.
First we prove that $f_n(x)$ converges pointwisely. For any $x \in (-1, 1)$, we know $|x|^n \to 0$ as $n \to \infty$. Since $-|x|^n \leq x^n \leq |x|^n$, by the Squeeze Theorem, $\lim f_n(x) = 0$.
Now to show x^n converges uniformly to 0, we find

$$\sup\{|x^n - 0| : x \in (-1, 1)\} = \sup\{x^n : x \in [0, 1)\} = 1 \nrightarrow 0 \text{ as } n \to \infty$$

Therefore, $f_n(x)$ does not converge uniformly to 0 on $(0, 1)$.

Example 24: Determine if $f_n(x) = \frac{nx}{1+n^2x}$ converges uniformly on $\mathbb{R}$ Let us find the limit function $f(x) = \lim f_n(x)$.

$$\lim_{n\to\infty} \frac{nx}{1 + n^2x} = \lim_{n\to\infty} \frac{\frac{nx}{n^2}}{\frac{1}{n^2} + \frac{n^2x}{n^2}} = 0$$

Now, we check for uniform convergence

$$\sup\left\{\frac{n|x|}{1 + n^2|x|^2}| : x \in \mathbb{R}\right\} \sim \sup\left\{\frac{t}{1 + t^2} : t \in \mathbb{R}\right\} = \frac{1}{2} \nrightarrow 0 \text{ as } t \to \infty$$

Therefore, $f_n(x)$ does not converge uniformly on $\mathbb{R}$.

Remark: This also could have been proved by finding the critical point x_0 as a function of n of $f_n(x)$ and showing $f_n(x_0) \to 0$. In this case, we get a contradiction because $f'(x) = 0$ when $n = 0$, which is never possible. So, f_n does not converge uniformly.

Theorem 4.2: Uniform Convergence and Continuity

If $\{f_n\}$ is a sequence of continuous functions on I and $f_n(x) \to f(x)$ as $n \to \infty$ uniformly on I then f is continuous on I.

Example 25: Let $f_n(x) = (1-|x|)^n$ on $[-1,1]$ and let $f(x) = 1$ if $x \neq 0$ and $f(0) = 0$. Determine whether $f_n(x) \to f(x)$ uniformly on $[-1,1]$ as $n \to \infty$.
Here we have $f_n(x)$ is continuous for all n on $[-1,1]$, but $f(x)$ contains a discontinuity at $x = 0$. By the above theorem, $f_n \nrightarrow f$ uniformly on $[-1,1]$. This could have also been solved using the definition of sup.

Theorem 4.3: Weierstrass M-Test

Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of functions on I such that

$$|f_n(x)| \leq M_n, x \in I$$

If $\sum_{n=1}^{\infty} M_n < \infty$, then $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly and absolutely on I.

Example 26: Show that $\sum_{n=1}^{\infty} \frac{\sin(nx)}{n^3 x}$ converges uniformly on $(0,\infty)$.
A useful fact that we will first prove is that $|\sin(nx)| \leq n|x|$. By the Mean Value Theorem, $\exists c \in (0,x)$ such that

$$\left|\frac{\sin(nx)-\sin(0)}{x-0}\right| = \left|\frac{\sin(nx)}{x}\right| = |n\cos(nx)| \iff \sin(nx) \leq n|x|$$

Now we use this fact to say

$$\sum_{n=1}^{\infty} \frac{\sin(nx)}{n^3 x} \leq \frac{1}{n^2}$$

Let $M_n = \frac{1}{n^2}$. By p-series test, $\sum M_n$ converges to a finite number. Therefore, by the M-test, $\sum_{n=1}^{\infty} \frac{\sin(nx)}{n^3 x}$ converges uniformly on $(0,\infty)$.

Example 27: Show that $\sum_{n=1}^{\infty} \frac{\sin(nx)}{n^2 x}$ does not converge uniformly on $(0,\infty)$.

Let us write the finite sum $\sum_{k=n+1}^{2n} \frac{\sin(kx)}{k^2 x}$ where $x = \frac{\pi}{4n}$ and $kx = \frac{k\pi}{4n}$. We know that $\frac{\pi}{4} kx \leq \frac{\pi}{2}$ for all k, n. So, $\sin(kx) \geq \frac{\sqrt{2}}{2}$. Lastly, we compute
$k^2 x = k(kx) \leq (2n)\left(2n + \frac{\pi}{4n}\right) = n\pi$. Therefore,

$$\sum_{k=n+1}^{2n} \frac{\sin(kx)}{k^2 x} \geq \sum_{k=n+1}^{2n} \frac{\frac{\sqrt{2}}{2}}{n\pi} \geq \frac{\sqrt{2}}{2\pi} = \epsilon_0 > 0$$

If we take $\epsilon_0 = \frac{\sqrt{2}}{2\pi}$ for any $n \in \mathbb{N}$, $x_n = \frac{\pi}{4n}$, we have

$$\left|\sum_{k=n+1}^{2n} \frac{\sin(kx)}{k^2 x}\right| \geq \epsilon_0.$$

So, by Cauchy Criterion, the series does not converge uniformly on $(0,\infty)$.

Example 28: Show that if $\sum |a_k| < \infty$, then $\sum a_k x^k$ converges uniformly on $[-1, 1]$ to a continuous function.
Because $x \in [-1, 1]$, $|a_k x^k| \leq |a_k|$. We know that $|a_k|$ converges and each partial sum is continuous because it is a polynomial. Therefore, by the M-test, $\sum |a_k|$ converges.

5 Weierstrass Approximations

Theorem 5.1: Weierstrass Approximation Theorem

Let $f \in C[0,1]$. Then, there are a sequence of polynomials $\{p_n\}_{n=1}^{\infty}$ such that $B_n f(x) \to f(x)$ uniformly as $n \to \infty$.

Example 29: Show that there does not exist a sequence of polynomials $\{p_n\}_{n=1}^{\infty}$ such that $p_n(x) \to \cos x$ as $n \to \infty$ uniformly on $\mathbb{R}$.

Suppose by contradiction there did exist such $\{p_n\}_{n=1}^{\infty}$. Then, there exists a polynomial such that

$$|p(x) - \cos x| < \frac{1}{2} = \epsilon \text{ for all } x \in \mathbb{R}$$

If $p(x)$ is a constant polynomial, we see an immediate contradiction because $\cos x$ is an oscillatory funciton and takes on multiple values, and so a constant function cannot converge to an oscillatory one. Now, say that p is not a constant polynomial. If p converges, then it must be bounded. So, we have

$$|p(x)| = |p(x) - \cos x| + |\cos x| < \frac{3}{2}$$

but $\lim_{x\to\infty} |p(x)| = \infty$, contradiction.

Example 30: Find a sequence of polynomials $\{p_n(x)\}_{n=1}^{\infty}$ converging to $\cos x$ uniformly on $[0, 2]$.

Let $f_0(x) = \cos(2x)$ on $[0, 1]$. We know that $f_0(x)$ is continuous on $[0, 1]$, therefore we can find a Bernstein polynomial $B_n(f_0(x))$ such that $B_n(f_0(x)) \to \cos(2x)$ uniformly. Let $g(x) = \frac{x}{2}$ on $[0, 2]$. Continuity is given for free because it is a polynomial, so we can find a Bernstein polynomial such that $B_n(g(x)) = B_n\left(\frac{x}{2}\right)$ converges uniformly to $\cos\left(2\left(\frac{x}{2}\right)\right) = \cos(2x)$ on $[0, 2]$.

6 Differentiation

Definition 6.1: Differentiation

Let $f(x)$ be a function on (a,b), $x_0 \in (a,b)$. We say that $f(x)$ is differentiable at x_0 if

$$\lim_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0} \text{ exists}$$

denoted by $f'(x)$.

Example 31: Let $f(x) = \begin{cases} x\sin\left(\frac{1}{x}\right), & x \neq 0 \\ 0, & x = 0 \end{cases}$. Find all $x_0 \in \mathbb{R}$ such that f is differentiable at x_0.

If $x_0 \neq 0$, then $\frac{1}{x}$ is differentiable at x_0 and so $\sin\left(\frac{1}{x}\right)$ is differentiable at x_0. Otherwise, for $x_0 = 0$, observe that

$$f'(x_0) = \lim_{x\to 0} \frac{x\sin\left(\frac{1}{x}\right) - 0}{x} = \sin\left(\frac{1}{x}\right)$$

and the limit does not exist, so f is differentiable for all $x \in \mathbb{R}$ except $x = 0$.

Example 32: Construct a function f on $\mathbb{R}$ that is continuous on $\mathbb{R}$ but nowhere differentiable on $\mathbb{R}$.

Let $f_0(x) = |x|, x \in [-1,1]$ and $f(x) = \sum_{n=1}^{\infty} \left(\frac{3}{4}\right)^n f_0(4^n x)$. By the M-test, it is continuous.

Example 33: Let $f(x) = x^2$ for x rational and $f(x) = 0$ for x irrational.

(a) Prove f is continuous at $x = 0$.

(b) Prove f is differentiable at $x = 0$.

(a) Let $\epsilon > 0$ and fix $\delta = \sqrt{\epsilon}$. Then, $|f(x) - f(0)| = |f(x)| \leq x^2$ for which $|x| < \delta \implies |x|^2 \leq \delta^2 = (\sqrt{\epsilon})^2 = \epsilon$.

(b) $\lim_{x\to 0} \left|\frac{f(x)-f(0)}{x-0}\right| = \left|\frac{f(x)}{x}\right| \leq \frac{x^2}{|x|} = |x| = 0$. So, $f'(0) = 0$ exists.

Example 34: Give an example that shows that if f_n is integrable on $[a,b]$ and $f_n \to f$ pointwisely on $[a,b]$ and f is integrable on $[a,b]$, then $\lim_{n\to\infty} \int_a^b f_n(x)dx \neq \int_a^b f(x)dx$.

Let $f_n(x) = \begin{cases} nx^{n-1}, & x \in [0,1) \\ 0, & x = 1 \end{cases}$

Theorem 6.2: Rolle's Theorem

Let $f(x)$ be continuous on $[a, b]$ and differentiable on (a, b) and $f(a) = f(b)$. Then, there exists a $c \in (a, b)$ such that $f'(c) = 0$.

Example 35: Let f and g be differentiable on an open interval I. Suppose a, b satisfy $a < b$ and $f(a) = f(b) = 0$. Show $f'(x) + f(x)g'(x) = 0$ for some $x \in (a, b)$.

Let $h(x) = f(x)e^{g(x)}$. Then, we have that $h(a) = h(b) = 0$ and $h'(x) = e^{g(x)}(f'(x) + f(x)g'(x))$. By Rolle's Theorem, there exists a $x \in (a, b)$ such that $h'(x) = 0$. We know that $e^{g(x)} > 0$ always, so we must have $f'(x) + f(x)g'(x) = 0$.

Theorem 6.3: Mean Value Theorem (MVT)

Let $f(x)$ be continuous on $[a, b]$ and differentiable on (a, b). Then, there exists a $c \in (a, b)$ such that

$$\frac{f(b) - f(a)}{b - a} = f'(c)$$

Remark: Rolle's Theroem is a corollary of the MVT

Example 36: Show that $|\cos x - \cos y| \leq |x - y|$ for all x, y.

Let $f(x) = \cos x$ on $[x, y]$. Since f is continuous on $[x, y]$ and differentiable on (x, y), we have that by the MVT, there exists a $c \in [x, y]$ such that

$$\left|\frac{\cos(x) - \cos(y)}{x - y}\right| = |-\sin(c)|$$

Because $|\sin c| \leq 1$, $|\cos x - \cos y| \leq |x - y|$ as desired.

Example 37: Let f be defined on $\mathbb{R}$ and suppose $|f(x) - f(y)| \leq (x - y)^2$ for all $x, y \in \mathbb{R}$. Prove f is a constant function.

We have that $\frac{|f(x)-f(y)|}{|x-y|} \leq |x - y|$ and $\lim_{x\to y} \frac{|f(x)-f(y)|}{|x-y|} \leq \lim_{x\to y} |x - y| \implies \lim_{x\to y} \frac{|f(x)-f(y)|}{|x-y|} \leq 0$. By MVT, this implies $|f'(y)| \leq 0$ and so $f'(y) = 0$, implying that f is constant.

Example 38: Show $ex \leq e^x$ for all $x \in \mathbb{R}$.

Let $g'(x) = e^x - e$ for all $x \in \mathbb{R}$. Then, we have that $g'(x) \leq 0$ for $x \leq 1$ and $g'(x) \geq 0$ for $x \geq 1$. Therefore, g has a local min at $x = 1$. We find $g(x) = e^x - ex$ and $g(1) = 0$. So, $g(x) \geq 0$ for all $x \in \mathbb{R} \implies e^x \geq ex$.

Example 39: Let $f(x) = x^2 \sin\left(\frac{1}{x}\right) + \frac{x}{2}$ for $x \neq 0$ and $f(0) = 0$. Show f is not increasing for on any open interval containing 0.

We show that for any open interval containing x_0, $\exists x_0$ such that $f'(x_0) \geq 0$. We have that

$$f'(x) = \left(-\frac{1}{x^2}\right) x^2 \cos\left(\frac{1}{x}\right) + 2x \sin\left(\frac{1}{x}\right) + \frac{1}{2} = \frac{1}{2} + 2x \sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right)$$

Let $x_n = \frac{1}{2\pi n}$ for $n \in \mathbb{N}$. Then, $f'(x_n) = -\frac{1}{2}$ and any open interval containing 0 also contains x_n. Therefore, f cannot be increasing.

Theorem 6.4: L'Hopital's Rule

Let $f(x) = \frac{g(x)}{h(x)}$ and suppose $\lim_{x \to a}$ yields an indeterminate form. Then,

$$\lim_{x \to a} f(x) = \lim_{x \to a} \frac{g'(x)}{h'(x)} = ... = \lim_{x \to a} \frac{g^{(n)}(x)}{h^{(n)}(x)}$$

Example 40: Find $\lim_{x\to\infty} \left(1 - \frac{1}{x}\right)^x$.

Let $L = \lim_{x\to\infty} \left(1 - \frac{1}{x}\right)^x$. Then, $\ln(L) = \lim_{x\to\infty} x \ln\left(1 - \frac{1}{x}\right)$. We have that $x = \frac{1}{\frac{1}{x}}$, so we rewrite it as

$$\lim_{x\to\infty} \frac{\ln\left(1 - \frac{1}{x}\right)}{\frac{1}{x}}$$

This is an indeterminate form $\frac{0}{0}$, so we can apply L'Hopital's Rule.

$$\lim_{x\to\infty} \frac{\frac{1}{x^2} \cdot \frac{1}{1-\frac{1}{x}}}{-\frac{1}{x^2}} = \lim_{x\to\infty} -\frac{1}{1 - \frac{1}{x}} = -1 = \ln(L) \implies L = e^{-1}$$

Example 41: Show $\lim_{x\to 0^+} \frac{1+\cos x}{e^x - 1} = +\infty$

Let $(x_n) \in (0, \infty)$ such that $x_n = \frac{1}{e^{x_n} - 1}$. Then $x_n > 0$ and $x_n \to 0$ and $1 + \cos(x_n) \to 2$ and $\lim_{x\to 0^+} \frac{2}{\text{small } +} = +\infty$.

Example 42: Find $\lim_{x\to\infty} x^{\sin\left(\frac{1}{x}\right)}$
$\ln L = \lim_{x\to\infty} \sin\left(\frac{1}{x}\right) \ln x = -\lim_{x\to\infty} \sin\frac{1}{x} \ln\frac{1}{x}$. Let $t = \frac{1}{x}$. Then, $t \to 0^+$. So we have

$$\lim_{t\to 0^+} \sin t \ln t = \lim_{t\to 0^+} \frac{\frac{\sin t}{t}}{\ln t} t = \lim_{t\to 0^+} \frac{t}{\ln t}$$

By L'Hopital's Rule, this gives

$$\lim_{t\to 0^+} t = 0 \implies \frac{1}{x} \to \infty \implies e^{\infty} = \infty$$

Example 43: Use L'Hopital's Rule to show that if f is twice differentiable in (a,b),

$$\lim_{h\to 0}\frac{f(x+h)+f(x-h)-2f(x)}{h^2}=f''(x)$$

Apply L'Hopital's Rule with respect to h.

$$\lim_{h\to 0}\frac{f'(x+h)-f'(x-h)}{2h}\implies \lim_{h\to 0}\frac{f''(x+h)+f''(x-h)}{2}=\lim_{h\to 0}\frac{2f''(x)}{2}=f''(x).$$

Example 44: Repeat Example 43 with the assumption that $f\in C^2([a,b])$.

Again, apply L'Hopital's Rule

$$\lim_{h\to 0}\frac{f'(x+h)-f'(x-h)}{2h}=\lim_{h\to 0}\frac{f'(x+h)-f'(x)+f'(x)-f'(x-h)}{2h}$$
$$=\frac{1}{2}f''(x)+\frac{1}{2}f''(x)=f''(x)$$

A similar argument can be made using the Mean Value Theorem.

Theorem 6.5: Taylor's Theorem

Let $f\in C^n([a,b])$. Then for any $x_0,x\in(a,b)$,

$$f(x)=\sum_{j=0}^{n-1}\frac{f^{(j)}(x_0)}{j!}(x-x_0)^j+\frac{f^{(n)}(x_0)}{(n)!}(x-x_0)^n$$

The last term is the remainder, or Lagrange error bound of the approximation. So, $f(x)=p_n(x)+R_n(x)$.

Example 45: Prove that $\cos x=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}x^{2n}$.
Let $f(x)=\cos x$. Then, for $n\geq 0, k\in\mathbb{Z}^+$

$$f^{(n)}(x)=\begin{cases}-\sin x & n=4k+1\\ -\cos x & n=4k+2\\ \sin x & n=4k+3\\ \cos x & n=4k\end{cases}\implies f^{(n)}(0)=\begin{cases}(-1)^{\frac{n}{2}} & n \text{ even}\\ 0 & n \text{ odd}\end{cases}$$

So, the Taylor Series for $\cos x=P_n(x)=\sum_{n=0}^{\infty}\frac{(-1)^{\frac{n}{2}}}{n!}x^n=\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k)!}x^{2k}$. Now, we prove the convergence of $P_n(x)\to\cos(x)$. Let $x\in\mathbb{R}$. Then, $\exists y\in(0,x)$ such that $|R_n(x)|=\left|\frac{f^{(n)}(y)}{n!}x^n\right|\leq\left|\frac{x^n}{n!}\right|\to 0$ as $n\to\infty$. Therefore, the Taylor Series for $\cos x$ converges for all $\mathbb{R}$.

Example 46: Prove the convergence of $p_n(x) := \sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)!} \to \sinh x$ for all $x \in \mathbb{R}$.
Let $y \in (0, x)$ be defined such that

$$R_n(x) = \left| \frac{f^{(n)}(y)}{n!} x^n \right|$$

We have that $|f^{(n)}(y)| = \left| \frac{e^y \pm e^{-y}}{2} \right| \leq \left| \frac{e^M + e^M}{2} \right| = e^M$. So,

$$R_n(x) \leq \left| \frac{e^M}{n!} x^n \right| \to 0$$

Therefore, we have convergence.

Example 47: Prove $(1+x)^\alpha = 1 + \sum_{n=1}^{\infty} \frac{\alpha \cdots (\alpha - n + 1)}{n!} x^n$.

Observe $f'(x) = \alpha(1+x)$ and $f^{(k)}(x) = \alpha(\alpha-1)\cdots(\alpha-k+1)(1+x)^{\alpha-k}$ and $f^{(k)}(0) = \alpha(\alpha-1)\cdots(\alpha-k+1)$. Therefore,

$$(1+x)^\alpha = 1 + \sum_{k=1}^{\infty} \frac{\alpha \cdots (\alpha - k + 1)}{k!} x^k.$$

Convergence follows a similar argument to Examples 45-46.

Example 48: Let $f(x)$ be twice differentiable on $[-3, 2]$ such that $f(0) = 0$ and $f(2) = f(-1) = 2$. Prove there is $x_0 \in (-1, 2)$ such that $f''(x_0) \geq 2$.

We have by Taylor's Theorem

$$\begin{aligned} f(2) &= f(0) + f'(0)(2-0) + \frac{(2-0)^2}{2!} f''(\xi_1) = f(0) + 2f'(0) + 2f''(\xi_1) = 2 \\ f(-1) &= f(0) + f'(0)(-1-0) + \frac{(-1)^2}{2!} f''(\xi_2) = f(0) - f'(0) + \frac{1}{2} f''(\xi_2) = 2 \end{aligned}$$

We do not know the value of $f'(0)$ is so that is what we want to eliminate. We get

$$2f''(\xi_1) + f''(\xi_2) = 6$$

Therefore $f''(\xi_1) \geq 2$ or $f''(\xi_2) \geq 2$.

Definition 6.6: Real Analytic Functions

Let $f(x)$ be a function on $I = (a, b)$. We say that f is real analytic on I if for any $x_0 \in I$, there is a $\delta > 0$ such that

$$f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

for $|x - x_0| < \delta$.

Example 49: Prove if f is a real analytic on (a, b) then $f \in C^\infty((a, b))$.

Let $f(x)$ be a real analytic function, then we know it converges to the power series $\sum_{n=0}^{\infty} a_n(x - x_0)^n$. We find that for a higher order derivative $f^{(k)}(x)$,

$$f^{(k)}(x) = \sum_{n=k}^{\infty} a_n \frac{n!}{(n-k)!}(x - x_0)^{n-k} = \sum_{j=0}^{\infty} a_j (x - x_0)^j \text{ for } j = n - k$$

can be differentiated term-by-term, which converges uniformly for any infinitesimally small subinterval $(-\delta, \delta)$ of (a, b) by the definition of R-A functions.

Example 50: Give an example of a C^∞ function that is not real analytic.

Let $f(x) = \begin{cases} e^{\frac{1}{x}} & x < 0 \\ 0 & x \geq 0 \end{cases}$. f is real analytic for $x < 0$, but not at $x_0 = 0$. Suppose by contradiction that f is real analytic at $x_0 = 0$, then

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n \qquad x \in (-\delta, \delta) \text{ for some } \delta > 0$$

$$f(x) = \sum_{n=0}^{\infty} = 0 \qquad x \in (-\delta, \delta)$$

Let $f(x) = e^{\frac{1}{x}}$. We reach a contradiction at $(-\delta, 0)$.

Definition 6.7: Convex Functions

Let f be a continuous function on $[a,b]$. We say that f is convex on $[a,b]$ if for any $x < y < b$ and $t \in (0,1)$, one has

$$f(tx+(1-t)y)) \leq tf(x)+(1-t)f(y)$$

Example 51: Let f and g be twice differentiable on (a,b). If g is convex increasing and f is convex on (a,b), then show that $(g \circ f)$ is convex on (a,b).

Fix $0 \leq t \leq 1, x, y \in (a,b)$. Then because f is convex and g increasing,

$$(g \circ f)(tx+(1-t)y) = g(f(tx-(1-t)y) \leq g(tf(x)+(1-t)f(y)) \leq t(g \circ f)(x)+(1-t)(g \circ f)(y)$$

which is convex by definition.

Example 52: Let f be twice differentiable and $f''(x) \geq 0$ on (a,b). Then, prove
(a) If $x_0 \in (a,b)$ is a critical point of f, then $f(x) \geq f(x_0)$, $x \in (a,b)$.

We have that by the Mean Value Theorem for $x > x_0, x_1 \in (x_0, x)$,
$f(x)-f(x_0) = f'(x_1)(x-x_0) = f'(x_1)(x-x_0) - f'(x_0)(x-x_0) =$
$(f'(x_1)-f'(x_0))(x-x_0) = f''(x_2)(x_1-x_0)(x-x_2)$ for $x_2 \in (x_0, x_1)$.

(b) For any $x_1, x_2 \in (a,b)$, one has $f\left(\frac{x_1+x_2}{2}\right) \leq \frac{f(x_1)+f(x_2)}{2}$

By the definition of convex functions (6.7), we get $f\left(\frac{x_1+x_2}{2}\right)$ if $t = \frac{1}{2}$. Following the definition completes the proof.

(c) For any $x_0 \in (a,b)$, one has $f(x) \geq f(x_0) + f'(x_0)(x-x_0)$

By the Mean Value Theorem, $\exists c \in (x_0, x)$ such that

$$\frac{f(x)-f(x_0)}{x-x_0} \leq f'(c)$$

Because $f''(x) \geq 0$, then f' is increasing and so $f'(x_0) \leq f'(c)$. Therefore,

$$\frac{f(x)-f(x_0)}{x-x_0} \geq f'(x_0) \Longrightarrow f(x) \geq f(x_0) + f'(x_0)(x-x_0)$$

Example 53: Let $p, q > 1$ satisfy $\frac{1}{p}+\frac{1}{q} = 1$. Then, prove $xy \leq \frac{x^p}{p}+\frac{y^q}{q}$ for any $x, y > 0$.

Let $f(x) = -\ln x$. Note that $f''(x) > 0$ for $(0,\infty)$ so it is a convex function. Thus, for any $1 < p, q < \infty$ satisfying $\frac{1}{p}+\frac{1}{q} = 1$, we use definition 6.7

$$f\left(\frac{x^p}{p}+\frac{y^q}{q}\right) \leq \frac{1}{p}f(x^p)+\frac{1}{q}f(y^q) \Longleftrightarrow xy \leq \frac{x^p}{p}+\frac{y^q}{q}$$

Example 54: If $x_1, x_2, \cdots, x_n$ are positive numbers then prove
$\sqrt[n]{x_1 x_2 \cdots x_n} \leq \frac{x_1+x_2+\cdots+x_n}{n}$.
Let $f(x) = -\ln(x)$, which is convex on $(0, \infty)$. Then the above inequality holds if and only if

$$-\ln\left(\sqrt[n]{x_1 x_2 \cdots x_n}\right) \leq -\frac{1}{n}\ln(x_1 + x_2 + \cdots + x_n).$$

The case in which $n = 2$ is clear. Suppose it holds for n; we will prove this also holds for $n+1$

$$-\ln\left(\frac{x_1 + \cdots + x_n + x_{n+1}}{n+1}\right) = -\ln\left(\frac{n}{n+1}\frac{x_1 + \cdots + x_n}{n} + \frac{x_{n+1}}{n+1}\right).$$

By definition of convexity, this is less than

$$\frac{n}{n+1}\left(-\ln\left(\frac{x_1 + \cdots + x_n}{n}\right)\right) + \frac{1}{n+1}(-\ln x_{n+1}) = \frac{1}{n+1}(-\ln x_1 - \ln x_2 - \cdots \ln x_{n+1})$$

By mathematical induction, the statement holds for all $n \in \mathbb{N}$.

7 Integration

Definition 7.1: Upper and Lower Darboux Sums

Given a partition P of $[a, b]$ and a bounded function f on $[a, b]$, define

$$M_i = \sup_{x\in[x_{i-1},x_i]} f(x) \qquad U(f;P) = \sum_{i=1}^{n} M_i(x_i - x_{i-1})$$

$$m_i = \inf_{x\in[x_{i-1},x_i]} f(x) \qquad L(f;P) = \sum_{i=1}^{n} m_i(x_i - x_{i-1})$$

as the lower and upper Darboux sums for f.

Theorem 7.2: Riemann Test for Integrability

Let f be a bounded function on $[a, b]$. Then f is integrable on $[a, b]$ if and only if for any $\epsilon > 0$, there exists a partition $P : a < x_0 < x_1 < \cdots < x_n = b$ such that

$$U(f;P) - L(f;P) < \epsilon$$

Example 55: Give an example of a function f on $[0, 1]$ which is not integrable on $[0, 1]$.

Let $D(x)$ be the Dirichlet function given by $D(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \in \mathbb{Q}^c \end{cases}$. Let $P = 0 = x_0 < x_1 < \cdots < x_n = 1$ be a partition of $[0, 1]$. Then, we find that $M_i = \sup_{x\in[x_{i-1},x_i]} D(x) = 1$ and $m_i = \inf_{x\in[x_{i-1},x_i]} D(x) = 0$. Then, $U(D;P) = 1 \neq L(D;P) = 0$. Therefore, $D(x)$ is not integrable on $[0, 1]$.

Example 56: Prove that any monotone functions are integrable on $[a, b]$.

Without loss of generality, suppose f is increasing on $[a, b]$. For any $\epsilon > 0$, choose N such that $\frac{(b-a)}{N}(f(b) - f(a)) < \epsilon$. Let $x_j = a + \frac{j}{N}(b - a)$ and P be the partition of $[a, b]$. Then, $M_j(f) = \sup_{x\in[x_{j-1},x_j]} f(x) \leq f(x_j)$ and $m_j(f) = \inf_{x\in[x_{j-1},x_j]} f(x) \geq f(x_j)$. Then,

$$U(f;P) - L(f;P) = \sum_{j=1}^{N}(M_j - m_j)(x_j - x_{j-1}) \leq \sum_{j=1}^{N}(f(x_j) - f(x_{j-1}))(x_j - x_{j-1})$$

Substitute $x_j - x_{j-1} = \frac{b-a}{N}$

$$= \sum_{j=1}^{N}(f(x_j) - f(x_{j-1}))\frac{b-a}{N} = \frac{b-a}{N}(f(x_N) - f(x_0)) = \frac{b-a}{N}(f(b) - f(a)) < \epsilon$$

So, f is integrable on $[a,b]$.

Example 57: If f and g are integrable as improper integrals on (a,b) where f and g may not be bounded on (a,b). Provide examples of f and g such that f and g are not integrable on (a,b).

$f = x^{\frac{1}{3}}, g = x^{\frac{2}{3}}$ on $(0,1)$.

Example 58: Construct a function on $[0,\infty)$ such that f is not integrable on $[0,b]$ for any $b>0$, f^2 is continuous on $[0,\infty)$, and $\int_0^\infty |f(x)|dx$ converges.

Define $f_0(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ -1 & x \in \mathbb{Q}^c \end{cases}$ and $f(x) = f_0(x)e^{-x}$. We show that f is not integrable on $[0,b]$ for $b>0$. Let P be a partition of $[a,b]$. Then,

$$U(f;P) - L(f;P) = \sum_{j=1}^{n}(M_j(f) - m_j(f))(x_j - x_{j-1}) = \sum_{j=1}^{n}(e^{-x_{j-1}} + e^{-x_{j-1}})(x_j - x_{j-1})$$

$$\geq 2e^{-b}\sum_{j=1}^{n}(x_j - x_{j-1}) = 2e^{-b}b > 0$$

Therefore, f is not integrable on $[a,b]$.

Example 59: Suppose f and g are integrable on $[a,b]$, then prove fg is also integrable on $[a,b]$.

The proof is quite a bit involved. The idea is to show f^2 is integrable first. If f is integrable, then it is bounded, say by B. Let $S \subseteq [a,b]$ such that for $x,y \in S$,

$$f^2(x) - f^2(y) = |(f(x)+f(y))(f(x)-f(y))| \leq 2B|f(x)-f(y)| \leq 2B(M(f;S) - m(f;S))$$

$$\implies M(f^2;S) - m(f^2;S) \leq 2B(M(f;S) - m(f;S))$$

Now, fix P as a partition of $[a,b]$. Then we see that the above inequality holds for any $S = [t_{k-1}, t_k]$. So, the difference of the upper and lower Darboux sums are

$$U(f^2;P) - L(f^2;P) = \sum_{k=1}^{n} M_k(f^2)(t_k - t_{k-1}) - \sum_{k=1}^{n} m_k(f^2)(t_k - t_{k-1}) \leq 2B\sum_{k=1}^{n}(t_k - t_{k-1})(M_k(f) - m_k(f))$$

$$= 2B(U(f;P) - L(f;P))$$

Let $\epsilon > 0$ be given such that

$$U(f;P) - L(f;P) < \frac{\epsilon}{2B} \implies U(f^2;P) - L(f^2;P) < \frac{\epsilon}{2B}(2B) = \epsilon$$

Therefore, f^2 is integrable. Now, note that for integrable functions f, g,

$4fg = (f+g)^2 - (f-g)^2$. The sum and scalar multiple of integrable functions are also integrable. This implies $(f+g)^2 - (f-g)^2$ are also integrable. Therefore, fg is integrable on $[a,b]$.

Theorem 7.3: Mean Value Theorem for Integrals

Let $f(x)$ be continuous on $[a,b]$, $g(x)$ be integrable on $[a,b]$ and $g(x) \geq 0$ on $[a,b]$. Then, $\exists x_0 \in [a,b]$ such that

$$f(x_0)\int_a^b g(x)dx = \int_a^b f(x)g(x)dx$$

Remark: If we let $g(x) = 1$, then this becomes

$$f(x_0) = \frac{1}{b-a}\int_a^b f(x)dx$$

Example 60: Prove $\exists x_0 \in (0,3)$ such that $\int_0^3 (x^2+2)\sin(x^2+1)dx = 15\sin(x_0^2+1)$.
Let $f(x) = \sin(x^2+1), g(x) = x^2+2$. Then, by the MVT for integrals, $\exists x_0 \in (0,3)$ such that

$$\sin(x_0^2+1)\int_0^3 (x^2+2)dx = 15\sin(x_0^2+1) = \int_0^3 (x^2+2)\sin(x^2+1)dx$$

Riemann-Stieltjes Integral

Example 61: Evaluate $\int_{-2}^{2} x^2 dF$ if

$$F(x) = \begin{cases} -1 & -2 < x \leq -1 \\ x+1 & -1 < x \leq 0 \\ x^2+1 & 0 < x \leq 1 \\ x^3+2 & x > 1 \end{cases}$$

Let $f(x) = x^2$. Then

$$\int_{-2}^{2} f(x)dF = \int_{-2}^{-1} fdF + f(-1)[F(-1^+) - F(-1^-)] + \int_{-1}^{0} fdF + f(0)[F(0^+) - F(0^-)]$$
$$+ \int_{0}^{1} fdF + f(1)[F(1^+) - F(1^-)] + \int_{1}^{2} fdF = \frac{643}{30}$$

Example 62: Show that $\int_0^\infty \frac{\sin x}{x} dx$ converges.

First, show that $\lim_{x\to 0} \frac{\sin x}{x} = 0$. Then, show $\int_1^\infty \frac{\sin x}{x}$ converges. You will need to do integration by parts you will get $\int_1^\infty \frac{\cos x}{x^2}$ which converges.

Example 63: Show that $\int_0^\infty \frac{\sin^2 x}{x(\ln(2+x))^p} dx$ converges for $p > 1$.

First show that $\lim_{x\to 0} \frac{\sin^2 x}{x(\ln(2+x))^p}$ exists. Then, compare to $\int_0^\infty \frac{1}{x(\ln(2+x))^p} dx$