Math 140A Main Concepts: Ryan Gomberg

1 Completeness of the Real Numbers

Definition 1.1: Natural Numbers and Induction

The set of natural numbers $\mathbb{N} := \{1, 2, 3, ...\}$. Natural numbers are commonly used in proofs involving mathematical induction, i.e. to show that a proposition is true for all $n \in \mathbb{N}$
(1) We first verify that the base case, P_1, is true.
(2) Verify that P_{n+1} is also true under the assumption that P_n is true.

Example 1: Show that the sum of natural numbers $1 + 2 + 3 + ... + n = \frac{n(n+1)}{2}$ for all $n \in \mathbb{N}$.
We first verify the base case, that is, $1 = \frac{1(1+1)}{2} = 1$.
Now, assume that $1 + ... + k = \frac{k(k+1)}{2}$ holds for all $k \leq n$ for $k, n \in \mathbb{N}$. Now, we find $1 + ... + (n + 1)$.

$$1 + ... + (n + 1) = 1 + ... + n + (n + 1)$$

By our assumption

$$1 + ... + n + (n + 1) = \frac{n(n + 1)}{2} + n + 1 = \frac{n^2 + n + 2n + 2}{2} = \frac{(n + 1)(n + 2)}{2}$$

which is exactly $\frac{n(n+1)}{2}$ when replaced by $n + 1$.

Example 2: Show that $11^n - 4^n$ is divisible by 7 when n is a positive integer.
First consider $n = 1$: $11 - 4 = 7$ which is divisible by 7.
Now assume that $11^k - 4^k = 7m$ for some $k, m \in \mathbb{N}$ and for $k \leq n$. We show that $11^{n+1} - 4^{n+1}$ is divisible by 7.

$$11^{n+1} - 4^{n+1} = 11^n(11) - 4^n(4) = 11^n(11) - 4^n(11 - 7) = 11(11^n - 4^n) - 7(4^n)$$

By our assumption, we reduce this to

$$11(7m) - 7(4^n) = 7(11 - 4^n)$$

which is divisible by 7.

Example 3: Show by induction that $2^n > n^2$ for $n \geq 5$.
We prove the base case ($n = 5$): $2^5 = 32 > 25 = 5^2$.
Let $n \in \mathbb{N}$ and assume the proposition holds. We claim for $n+1$,

$$2^{n+1} > (n+1)^2 \implies 2^n \cdot 2 > 2n^2 \implies 2 > \left(\frac{n+1}{n}\right)^2$$

For $n \geq 5, 2 > \frac{6^2}{5^2} = \frac{36}{25}$. So, $(n+1)^2 < 2n^2$. Now we have

$$2^{n+1} = 2 \cdot 2^n > 2n^2 > (n+1)^2$$

by our inductive hypothesis.

Definition 1.2: Integers and Rational Numbers

The set of integers $\mathbb{Z} := \{0, \pm 1, \pm 2, \pm 3, ...\}$
The set of rational numbers $\mathbb{Q} := \{\frac{m}{n} | m, n \in \mathbb{Z}, n \neq 0\}$

Theorem 1.3: Rational Root Test

Suppose $c_0, c_1, ..., c_n \in \mathbb{Z}$ such that $c_0, c_n \neq 0$ and $r = \frac{c}{d} \in \mathbb{Q}$ and

$$c_n r^n + c_{n-1} r^{n-1} + ... + c_1 r + c_0 = 0.$$

Then c divides c_0 and d divides c_n.

Theorem 1.4: Properties of $\mathbb{R}$: Supremum and Infimum

First, let $S \subset R$ be a non-empty set. We say the upper bound of S is y if $\exists y \in \mathbb{R}$ such that $x \leq y \; \forall x \in S$. On the contrary, the lower bound of S is z if $\exists z \in \mathbb{R}$ such that $x \geq z \; \forall x \in S$. We use these to define the following:

We say $\sup S$ is the **supremum**, or least upper bound given that y is an upper bound and for $x < y$, x is NOT an upper bound of S.

Likewise, $\inf S$ is the **infimum**, or greatest lower bound.

Example 4: Consider the set $S = \{-1, -\frac{1}{2}, ..., -\frac{1}{n}\}$. Then $\sup S = -\frac{1}{n}$ and $\inf S = -1$.

Example 5: Consider the set $S = \left\{\bigcap_{n=1}^{\infty} \left[-\frac{1}{n}, 1 + \frac{1}{n}\right]\right\}$. Then $\sup S = \sup([0,1]) = 1$.

Example 6: Let S, T be nonempty, bounded subsets of $\mathbb{R}$. Prove that if $S \subseteq T$, then $\inf T \leq \inf S \leq supS \leq supT$.

(1) $\inf T \leq \inf S$. We have that $\inf T \leq T \forall t \in T$ and by $S \subseteq T$, $\inf T \leq s\ \forall s \in S$. Hence we have that $\inf T$ is a lower bound for S, and so $\inf T \leq \inf S$.
(2) $\inf S \leq supS$. This is a given statement by the definition of $\inf S$ and $\inf T$.
(3) $supS \leq supT$. Because $S \subseteq T$, it follows that $\sup T \geq s\ \forall s \in S$. Therefore $\sup T$ is the lowest upper bound for S, T, and so $\sup S \leq supT$.
By combining the above inequalities, we complete the proof.

Definition 1.5: Archimedean Property

If $a, b \in \mathbb{R}, a > 0$, then $na > b$ for some $n \in \mathbb{N}$.

Theorem 1.6: Denseness of $\mathbb{Q}$ in $\mathbb{R}$

If $a < b \in \mathbb{R}$, then $\exists r \in \mathbb{R}$ such that $a < r < b$.

Example 7: Prove that if $a > 0$, then there exists $n \in \mathbb{N}$ such that $\frac{1}{n} < a < n$.
We use the Archimedean property. For $nx > y$, $x, y \in \mathbb{R}$, let $x = 1$ and $y = a$. Then $\exists n_1 \in \mathbb{N}$ such that $n > a$. Now, $x = a$ and $y = 1$. Then, $\exists n_2 \in \mathbb{N}$ such that $n_2 a > 1 \rightarrow a > \frac{1}{n_2}$.
Consider $n = \max\{n_1, n_2\} \implies n \geq n_1, n \geq n_2$. So, $a < n_1 \geq n$ and $\frac{1}{n} \leq \frac{1}{n_2} < a$. We then obtain the inequality

$$\frac{1}{n} \leq \frac{1}{n_2} < a < \frac{1}{n} < n \implies \frac{1}{n} < a < n$$

so such $n \in \mathbb{N}$ exists.

Example 8: Consider a, b where $a < b$. Use denseness of $\mathbb{Q}$ to show there are infinitely many rationals between a and b.
We proceed with a proof by contradiction. Fix an interval (a, b) for $a, b \in \mathbb{R}$ and assume a finite number of rationals, which we will denote as $R = \{r_1, r_2, ..., r_n\}$ within (a, b). Because $|R|$ is finite, $\exists M = \max R$ such that $a < M < b$. However, $\exists r \in \mathbb{Q}$ such that $M < r < b$ by the denseness of $\mathbb{Q}$, but $M < r$ and M is the largest rational number in (a, b), thus a contradiction.

Example 9: Let $a, b \in \mathbb{R}$. Show if $a \leq b + \frac{1}{n}\ \forall n \in \mathbb{N}$, then $a \leq b$.
We proceed by contradiction. Suppose $a \geq b + \frac{1}{n}$ and $a > b$. Then $a - b > 0$ and $b + \frac{1}{n} < a$ but by our assumption, $b + \frac{1}{n} \geq a$. Contradiction.

2 Limits and Sequences

We first introduce the definition of a sequence:

Definition 2.1: Sequences

A sequence of real numbers is a map from $\mathbb{N} \to \mathbb{R}$ denoted by $\{a_n\}_{n=1}^{\infty}$ or simply $\{a_n\}$.

Example 10: The sequence $s_n = (-1)^n$ has alternating terms -1, 1, -1, 1.

Example 11: The first 4 terms of the sequence $s_n = \frac{4n^3+3n}{n^3-6}$ are $-\frac{7}{5}, 19, \frac{39}{7}, \frac{134}{29}$. Later, we will show that this sequence converges to 4.

Definition 2.2: "$\epsilon - N$" Definition of a Limit

We say a sequence $\{a_n\}$ converges to $a \in \mathbb{R}$ if for each $\epsilon > 0$, $\exists N > 0$ such that $|a - a_n| < \epsilon$ for all $n > N$. If such a does not exist, we say $\{a_n\}$ diverges.

Remark: Finding such N is written as a function of ϵ. Also note that N does not have to be an integer, so long as the limit converges for any $n > N$, then the proof works.

Example 12: Show that $\lim_{n\to\infty} \frac{1}{n^2} = 0$ using the $\epsilon - N$ definition.

Let $\epsilon > 0$ be given such that $\forall\ n > N$,

$$\left|\frac{1}{n^2} - 0\right| < \epsilon.$$

Choose $N = \frac{1}{\sqrt{\epsilon}}$, then

$$\left|\frac{1}{n^2} - 0\right| < \frac{1}{N^2} = \epsilon$$

as desired. This completes the proof.

Example 13: Show that $\lim_{n\to\infty} \frac{4n^3+3n}{n^3-6} = 4$.

Let us find N such that for $\epsilon > 0$ and $\forall\, n > N$,

$$\left|\frac{4n^3+3n}{n^3-6} - 4\right| = \left|\frac{3n+24}{n^3-6}\right| < \epsilon$$

Directly finding n would be very tedious, so instead we make some assumptions that will make the computation easier. We know that for $n \geq 1$, $3n + 24 \leq 27n$ and for $n \geq 3$, $n^3 - 6 \geq \frac{1}{2}n^3$. We can know simplify the above inequality to

$$\left|\frac{27n}{\frac{n^3}{2}}\right| = \left|\frac{54}{n^2}\right| < \epsilon$$

Therefore, one possibility is to choose $N = \sqrt{\frac{54}{\epsilon}}$. However, this does not finish the proof as we have multiple choices for $n > N$ that could satisfy the inequality. We have $n \geq 1, n \geq 3, n \geq \sqrt{\frac{54}{\epsilon}}$. We want the largest N possible, so let $N = \max\left\{3, \sqrt{\frac{54}{\epsilon}}\right\}$. Now we are done.

Example 14: Show that $\lim \sqrt{s_n} = \sqrt{s}$ for a sequence s_n.

We have to analyze two cases: One where $\lim s_n = 0$ and $\lim s_n > 0$.

(1) $\lim s_n = 0$: We have $|\sqrt{s_n} - 0| < \epsilon$. Let $N = \epsilon^2$ and we are done.

(2) $\lim s_n > 0$: We have $|\sqrt{s_n} - \sqrt{s}| < \epsilon$. We can "irrationalize" the denominator to get

$$|\sqrt{s_n} - \sqrt{s}| \left| \frac{\sqrt{s_n} + \sqrt{s}}{\sqrt{s_n} + \sqrt{s}} \right| = \left| \frac{s_n - s}{\sqrt{s_n} + \sqrt{s}} \right|$$

We say

$$\frac{1}{\sqrt{s_n} + \sqrt{s}} < \frac{1}{\sqrt{s}} \implies \left| \frac{s_n - s}{\sqrt{s}} \right| < \epsilon \implies |\sqrt{s_n} - \sqrt{s}| < \sqrt{s}\epsilon$$

Therefore setting $N = \sqrt{s}\epsilon$ completes the proof.

Definition 2.3: Bounded Sequence

We say a sequence $\{s_n\}$ is bounded below and above if there exists $M > 0$ such that $|s_n| \leq M$ for all $n \in \mathbb{N}$.

Theorem 2.4: Bounded Sequences and Convergence

All convergent sequences are bounded.

Example 15: Let (s_n) be a sequence that converges. Show that (1) if $s_n \geq a$ for all but finitely many n, then $\lim s_n \geq a$, (2) if $s_n \leq b$ for all but finitely many n, then $\lim s_n \leq b$, and (3) if $s_n \in [a, b]$ for all but finitely many n, then $\lim s_n$ belongs to $[a, b]$.

(1) We have that $|s_n - s| = |s - s_n| < \epsilon \implies -\epsilon < s - s_n < \epsilon$. We can ignore the right side of the inequality and say $\epsilon + s < s_n$. The fact $s_n \geq a$ implies $s > a - \epsilon$ for some $n > N$.

(2) We use the right side of the inequality from (1) and say $s < \epsilon + s_n$. The fact $s_n \leq b$ implies $s < b + \epsilon$ for some $n > N$.

(3) Parts (1) and (2) imply that s_n is bounded below by b and above by a. So, s_n is bounded by $[a, b]$ for all but finitely many n.

Lemma 2.5: Squeeze Lemma

Suppose a_n, b_n, c_n are sequences such that $a_n \leq b_n \leq c_n \forall n$. Then, $\lim a_n = \lim c_n = s \in \mathbb{R} \implies \lim b_n = s$.

Example 16: Given sequences $a_n = -\frac{1}{n}\sin\left(\frac{1}{n^2}\right), b_n = \sin\left(\frac{1}{n^2}\right), c_n = \frac{1}{n}\sin\left(\frac{1}{n^2}\right)$, we have that $\lim b_n = \lim a_n = \lim c_n = 0$.

Lemma 2.6: Limits of Special Sequences

(1) For $p > 0$, $\lim_{n\to\infty} n^{-p} = 0$.
(2) For $|a| < 1$, $\lim_{n\to\infty} a^n = 0$.
(3) $\lim_{n\to\infty} n^{\frac{1}{n}} = 1$.
(4) For $a > 0$, $\lim_{n\to\infty} a^{\frac{1}{n}} = 1$.

Definition 2.7: "M-N" definition for Infinite Limits

We say $\lim s_n = +\infty$ provided that for each $M > 0$, $\exists N > 0$ such that for all $n > n$, $s_n > M$.

This time, we want to find N as an inequality with respect to M.

Example 17: Show that $\lim(\sqrt{n} - 1) = +\infty$.
Let $M > 0$ be given, and choose $N = (M+1)^2$. Therefore, we have

$$\sqrt{n} - 1 > (M+1) - 1 = M.$$

Example 18: Show that $\lim \frac{n^2+5}{n+1} = +\infty$.
We adapt a similar approach compared to Example 4. Let us propose $n^2 + 5 > n^2$, $n + 1 \leq 2n$ for all $n \geq 1$. So, we have that

$$\left|\frac{n^2+5}{n+1}\right| > \left|\frac{n^2}{2n}\right| = \left|\frac{n}{2}\right|$$

Therefore, let $N = 2M$. Then $n > N$ implies

$$\frac{n^2+5}{n+1} > \frac{n^2}{2n} = \frac{n}{2} > M$$

This completes the proof.

Example 19: Suppose s_n, t_n satisfy $\lim s_n = \infty, \lim t_n > 0$. Prove $\lim s_n t_n = \infty$.

Because $\lim t_n > 0$, $\exists m_0 \in \mathbb{R}$ such that $t_n > m_0$ for all $n > N$.
As for s_n, $\exists M \in \mathbb{R}$ such that $s_n > \frac{M}{m_0}$ for all $n > N'$.
Now, let $N^1 = \max\{N, N'\}$. Then $n > N^1$ implies $s_n t_n > \frac{M}{m_0} \cdot m_0 = M$, as desired.

Definition 2.8: Monotone Sequences

A **monotone sequence** is a strictly increasing or decreasing sequence. More precisely, for a sequence s_n,
(1) $s_{n+1} < s_n$ for all n yields a monotonically decreasing sequence.
(2) $s_{n+1} > s_n$ for all n yields a monotonically increasing sequence.

Example 20: The sequence $s_{n+1} = s_n - 1, s_1 = 5$ is a monotonically decreasing sequence.

Example 21: The sequence $s_n = 5 - \frac{1}{n}$ is a monotonically increasing sequence. In fact, this sequence converges to 5.

Theorem 2.9: Bounded and Monotone Sequences

All bounded, monotone sequences converge.

Example 22: Show that the sequence $s_n = \frac{s_{n-1}^2+5}{2s_{n-1}}$, $s_1 = 5$, converges and find its limit.

We use the fact that $a^2 + b^2 > 2ab \implies s_{n+1} + \sqrt{5} \geq 2s_{n-1}\sqrt{5}$. So

$$\frac{s_{n-1}^2 + 5}{2s_{n-1}} \geq \frac{2s_{n-1}\sqrt{5}}{2s_{n-1}} \geq \sqrt{5}$$

So $s_n \geq \sqrt{5}\ \forall n$. To show that it is decreasing, we show that $s_n - s_{n-1} < 0$. We use the recursively defined sequence

$$s_n - s_{n-1} = \frac{s_{n-1}^2 + 5}{2s_{n-1}} - s_{n-1} = \frac{5 - s_{n-1}^2}{2s_{n-1}}$$

We already proved $s_n \geq \sqrt{5}$, so s_n is monotonically decreasing and is bounded above 5. Since $s_n \geq \sqrt{5}$ and $s_n \leq 5$, we conclude that s_n is bounded. In addition, because it is a monotone sequence, we know that s_n must converge by the above theorem. To find its limit, we use the fact that $\lim_{n\to\infty} s_n = s$ to obtain

$$2s^2 = s^2 + 5 \implies s = \sqrt{5}.$$

Definition 2.10: Limsup and Liminf

Let s_n be a sequence and define two related sequences: u_N, v_N.
(1) $u_N := \sup\{s_n : n > N\}$
(2) $v_N := \inf\{s_n : n > N\}$
The **limit superior**, $\limsup s_n$, is defined as

$$\begin{cases} \lim_{n\to\infty} v_N & \text{if } s_n \text{ bounded above} \\ \infty & \text{if } s_n \text{ unbounded above} \end{cases}$$

The **limit inferior**, $\liminf s_n$, is defined as

$$\begin{cases} \lim_{n\to\infty} u_N & \text{if } s_n \text{ bounded below} \\ -\infty & \text{if } s_n \text{ unbounded below} \end{cases}$$

Example 23: Let us compare two sequences: $s_n = \sin\left(\frac{n\pi}{2}\right)$, $t_n = \sin\left(\frac{n\pi}{3}\right)$.
We know that s_n is one of 3 values $\{-1, 0, 1\}$ and oscillates between -1 and 1. So, $\limsup s_n = 1, \liminf s_n = -1$. This means that as n can get larger and larger but s_n never goes above 1 and below -1.
As for t_n, it is one of 3 values $\left\{-\frac{\sqrt{3}}{2}, 0, \frac{\sqrt{3}}{2}\right\}$. By the same logic, $\limsup t_n = \frac{\sqrt{3}}{2}, \liminf t_n = -\frac{\sqrt{3}}{2}$.

Example 24: Let $s_n = 6 + (-1)^n\left(1 + \frac{5}{n}\right)$.

For even values of n, we have a monotonically decreasing sequence, converging to 7. For odd values of n, we have a monotonically increasing sequence, converging to 5. So, $\limsup s_n = 7$, $\liminf s_n = 5$. Here, we can consider the sequence $v_N = s_{2k}$ and $u_N = s_{2k-1}$.

Example 25: Prove that $\limsup |s_n| = 0 \implies \lim s_n = 0$
We have that $|s_n| \geq 0$, so $\liminf |s_n| \geq 0$. Therefore, we have the following inequality

$$0 \leq \liminf |s_n| \leq \limsup |s_n| = 0$$

Therefore $\limsup |s_n| = \liminf |s_n| \iff \lim |s_n| = 0$. By $\lim |s_n| = 0, \exists \epsilon > 0$ such that for all $n > N$,

$$||s_n| - 0| < \epsilon \implies |s_n| < \epsilon.$$

We are done.

Theorem 2.11: Liminf/Limsup/Lim

Suppose $\lim s_n$ exists. Then $\liminf s_n = \limsup s_n = \lim s_n = s$.

Definition 2.12: Cauchy Sequences and Convergence

Recall the $\epsilon - \delta$ definition of limits. A sequence s_n is Cauchy if for each $\epsilon > 0, \exists N > 0$ such that $|S_n - S_m| < \epsilon$ for all $m, n > N$.

A sequence is convergent iff it is Cauchy

Example 26: Prove that the sequence $s_n = \frac{1}{n^3}$ converges using the notion of Cauchy sequences.

Let $N = \frac{1}{\sqrt[3]{\epsilon}}$. Then for all $m, n > N$ and for some $\epsilon > 0$, we have that

$$\left| \frac{1}{n^3} - \frac{1}{m^3} \right| < \frac{1}{n^3} < \frac{1}{N^3} < \epsilon.$$

Example 27: Let s_n be a sequence that satisfies $|s_{n+1} - s_n| < 3^{-n}$. Show that s_n is a convergent sequence and therefore a Cauchy sequence.

Let us choose m such that $m > n$. We take the inequality

$$|s_m - s_n| = |s_m - s_{m-1} + s_{m-1} - s_{m-2} + ... + s_{n+1} - s_n|$$

Separating each pair yields

$$|s_m - s_n| = |s_m - s_{m-1}| + |s_{m-1} - s_{m-2}| + ... + |s_{n+1} - s_n|$$

By our assumption we have

$$|s_m - s_{m-1}| + |s_{m-1} - s_{m-2}| + ... + |s_{n+1} - s_n| < \frac{1}{3^{m-1}} + \frac{1}{3^{m-2}} + ... + \frac{1}{3^n}.$$

By $m > n$, the terms are increasing, so

$$\frac{1}{3^{m-1}} + \frac{1}{3^{m-2}} + ... + \frac{1}{3^n} < \frac{1}{3^n} + \frac{1}{3^n} + ... + \frac{1}{3^n} = \frac{k}{3^n}$$

for some $k \in \mathbb{N}$. Therefore $|s_m - s_n| < \frac{k}{3^n}$. To show that it is Cauchy, let $\epsilon > 0$ be given and so for all $m, n > N$

$$\left| \frac{k}{3^n} - 0 \right| < \epsilon$$

Setting $N = \log_3 \left(\frac{k}{\epsilon} \right)$ completes the proof. Therefore, s_n is a convergent sequence.

Definition 2.13: Subsequences

Let (s_n) be a sequence. A subsequence (s_{n_k}) is a subset $(s_{n_k}) \subseteq (s_n)$ where

$$n_1 < n_2 < n_3 < \ldots$$

Subsequences are infinite subsets, whose order is inherited from the main sequence.

Example 28: Consider $s_n = \frac{(-1)^n}{n}$. We can extract two main subsequences: one when n is even $(n = 2k)$ and for when n is odd $(n = 2k - 1)$ for $k \in \mathbb{N}$. So

$$s_n = \begin{cases} s_{2k-1} = -\frac{1}{2k-1} & n \text{ odd} \\ s_{2k} = \frac{1}{2k} & n \text{ even} \end{cases}$$

Example 29: The sequence $\sin(n)$ has infinitely many subsequences because it is an oscillating funciton. For example, the subsequence s_{n_k} where $n = \frac{\pi}{2} + 2\pi k$ only returns 1.

Theorem 2.14: Bolzano-Weierstraβ Theorem

Every bounded sequence has a convergent subsequence.

Example 30: We can refer to Examples 28 and 29. Let us look at the subsequence s_{2k} for Example 28. The subsequence $\frac{1}{2k}$ is bounded by $\left(0, \frac{1}{2}\right]$, and so it converges, namely to 0. For example 29, the subsequence $s_{\frac{\pi}{2}+2\pi k}$ is bounded by 1, so obviously it converges to 1.

Example 31: Find a convergent subsequence for $s_n = n(1 + (-1)^n)$
We have that $(1 + (-1)^n) = 0$ whenever n is odd. So, the subsequence $s_{2k-1} = 0$ is a convergent subsequence.

Theorem 2.15: Properties of $\limsup$ and $\liminf$

Let s_n, t_n be two sequences.
(1) $\limsup(s_n + t_n) \leq \limsup s_n + \limsup t_n$. If one of s_n, t_n converge to a real number, then we have equality.
(2)

$$\liminf \left|\frac{s_{n+1}}{s_n}\right| \leq \liminf |s_n|^{\frac{1}{n}} \leq \limsup |s_n|^{\frac{1}{n}} \leq \limsup \left|\frac{s_{n+1}}{s_n}\right|$$

If $\lim \left|\frac{s_{n+1}}{s_n}\right| = L$ then $\lim |s_n|^{\frac{1}{n}} = L$.

Example 32: As an example of why $\limsup(s_n + t_n) \neq \limsup s_n + \limsup t_n$ in some cases, consider $s_n = (-1)^n, t_n = (-1)^{n+1}$. $\limsup(s_n + t_n) = 0$ but $\limsup(s_n) + \limsup(t_n) = 2$.

3 Series

Before we introduce the definition of a series, let us define a partial sum.

Definition 3.1: Partial Sum

Let a sequence $(a_n)_{n=1}^{\infty}$ be given. Then, the n^{th} partial sum of s_n is defined as

$$s_n := \sum_{k=m}^{n} a_k = a_m + a_{m+1} + ... + a_m$$

Example 33: The 5th partial sum of the sequence $\frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25}$.

Definition 3.2: Series and Convergence

A series is an infinite limit case of a partial sum, where $n \to \infty$, or more precisely $\sum_{n=m}^{\infty} a_n$.
(1) A series converges (to s or $\pm\infty$) or diverges by oscillation, as does the sequence s_n (but not to the same number!).
(2) A series **converges absolutely** if $\sum |a_n|$ converges.
(3) A series **converges conditionally** if $\sum a_n$ converges, but not absolutely.
(4) (**Divergence Test**) If $\sum a_n$ converges, then $\lim a_n = 0$, but the converse is not always true.

Example 34: We will see that the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges conditionally, meaning that $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges but $\sum_{n=1}^{\infty} \left|\frac{(-1)^n}{n}\right|$ does not.

Theorem 3.3: Properties of Series

Convergent series preserve linear combinations of series. However, the product of two series are generally not preserved.

We now look at various convergence tests and definitions.

Definition 3.4: Geometric Series

A **geometric series** is of the following form

$$\sum_{n=m}^{\infty} ar^n \begin{cases} \text{converges to } \frac{ar^m}{1-r} & \text{if } |r| < 1 \\ \text{diverges to } \infty & \text{if } r \geq 1 \\ \text{diverges by oscillation} & \text{if } r \leq -1 \end{cases}$$

In the case $m = 1$ and $|r| < 1$, the geometric series converges to $\frac{a}{1-r}$.

Example 35: The geometric series $\sum_{n=-1}^{\infty} 2\left(-\frac{1}{2}\right)^n$ converges to

$$\frac{2\left(-\frac{1}{2}\right)^{-1}}{1+\frac{1}{2}} = -\frac{8}{3}.$$

Definition 3.5: $p-$series

A $p-$**series** is of the following form

$$\sum_{n=1}^{\infty} \frac{k}{n^p} \begin{cases} \text{converges} & \text{if } p > 1 \\ \text{diverges} & \text{if } p \leq 1 \end{cases}$$

Finding for what value a p-series converges to is not as straightforward compared to a geometric series.

Theorem 3.6: Cauchy Criterion for Series

A series $\sum a_n$ converges if and only if

$$\forall \epsilon > 0, \exists N \text{ such that } m \geq n > N \implies |s_m - s_{n-1}| = \left|\sum_{k=n}^{m} a_k\right| < \epsilon.$$

Example 36: Show that the $p-$series $\frac{1}{n}$ diverges by Cauchy's criterion.
Assume that the series converges and let $\epsilon = \frac{1}{2}$. Then $\exists N$ such that $m \geq n > N \implies \left|\sum_{k=n}^{m} \frac{1}{k}\right| < \frac{1}{2}$. Fix $m = 2(n-1) \geq n$, then

$$\frac{1}{2} > \left|\sum_{k=n}^{m} \frac{1}{k}\right| = \left|\frac{1}{n} + \ldots + \frac{1}{m}\right| \geq \frac{m-(n-1)}{m} = 1 - \frac{n-1}{m} = \frac{1}{2}$$

and $\frac{1}{2} > \frac{1}{2}$ is obviously not true, so we reach a contradiction.

Theorem 3.7: Comparison Test

Suppose $a_n \geq 0$ for all n. Then,
(1) If $\sum a_n$ converges and $|b_n| \leq |a_n|$ $\forall n$, then $\sum b_n$ converges.
(2) If $\sum a_n$ diverges to ∞ and $b_n \geq a_n$ $\forall n$, then $\sum b_n$ diverges.

Example 37: Show that the series $\sum_{n=1}^{\infty} \frac{n}{n^2+3}$ diverges.
We have that $n^2 + 3 \leq 4n^2$ for $n \geq 1$ and so $\frac{n}{n^2+3} \geq \frac{n}{4n^2} = \frac{1}{4n} = \frac{1}{4} \cdot \frac{1}{n}$, and we know that by the $p-$series test, $\sum \frac{1}{n}$ diverges. So, by comparison to $\sum \frac{1}{4n}$, $\sum \frac{n}{n^2+3}$ also diverges.

Example 38: Show that the series $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ converges.
We have that $n^2 + 1 > n^2$ for $n \geq 1$ and so $\frac{1}{n^2+1} \leq \frac{1}{n^2}$. We know that $\sum \frac{1}{n^2}$ converges by $p-$series, so $\frac{1}{n^2+1}$ converges by comparison to $\frac{1}{n^2}$.

Example 39: Show that the series $\sum_{n=1}^{\infty} \frac{2n+1}{(n+2)3^n}$ converges.

We know that $\frac{2n+1}{(n+2)3^n} \leq \frac{2}{3^n}$, where $\sum \frac{2}{3^n}$ is a geometric series that converges to $\frac{2(\frac{1}{3})}{1-\frac{1}{3}} = 1$. Therefore, by comparison with $\sum \frac{2}{3^n}$, the series $\sum_{n=1}^{\infty} \frac{2n+1}{(n+2)3^n}$ converges, more precisely, to a value ≤ 1.
Example 40: Show that the series $\sum_{n=1}^{\infty} \frac{(n^2+1)^{\frac{1}{2}}}{(1+\sqrt{n})^4}$ diverges.
We have that $(n^2 + 1)^{\frac{1}{2}} \geq n$ for $n \geq 1$ and $(1 + \sqrt{n})^4 \leq (2\sqrt{n})^4$ for $n \geq 1$. Because $(2\sqrt{n})^4$ is larger than n for all $n \geq 1$, then we show that

$$\frac{(n^2+1)^{\frac{1}{2}}}{(1+\sqrt{n})^4} > \frac{1}{16n}$$

and $\sum \frac{1}{16n}$ is a $p-$series that diverges, so $\sum \frac{(n^2+1)^{\frac{1}{2}}}{(1+\sqrt{n})^4}$ diverges.

Theorem 3.8: Root Test

Let $\alpha = \limsup_n |a_n|^{\frac{1}{n}}$. The series $\sum_{n=1}^{\infty} a_n$
(1) converges absolutely if $\alpha < 1$.
(2) diverges if $\alpha > 1$.
Note that for $\alpha = 1$, the test is inconclusive.

Theorem 3.9: Ratio Test

Let $\sum_{n=1}^{\infty} a_n$ be a series of nonzero terms. Then the series
(1) converges absolutely if $\limsup \left|\frac{a_{n+1}}{a_n}\right| < 1$.
(2) diverges if $\liminf \left|\frac{a_{n+1}}{a_n}\right| > 1$.
Note that if

$$\liminf \left|\frac{a_{n+1}}{a_n}\right| \leq 1 \leq \limsup \left|\frac{a_{n+1}}{a_n}\right|$$

then the test is inconclusive.

Remark: The ratio test is a weaker case of the root rest. The ratio test works well when a series contains factorial or exponential terms. Otherwise, it is better to use the Root Test.

4 Continuity

Pointwise Continuity

We first start with the definition of real-valued function and domain/image. Given a function $f(x) : x \to \mathbb{R}$,

We call the domain of a real-valued function $\text{Dom}(f) = \mathbb{N}$. In general, the domain is $\text{Dom}(f) = (a, b), [a, b], (a, b], [a, b)$

The image, or range, is $\text{Im}(f) = f(I) = \{f(I) : x \in I\}$.
This definition helps us introduce the concept of pointwise continuity, as shown below.

Definition 4.1: Pointwise Continuity

We say f is **pointwise continuous** at $x = x_0 \in \text{Dom}(f)$ provided that for each $\epsilon > 0, \exists \delta > 0$ such that $|f(x) - f(x_0)| < \epsilon$ for all $x \in \text{Dom}(f)$ with $|x - x_0| < \delta$.

The idea is that δ depends on ϵ, so we want to find such δ as a function of ϵ.

Example 41: Show that $f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right), & x \neq 0 \\ 0, & x = 0 \end{cases}$ is pointwise continuous at $x_0 = 0$.

The first thing we notice is that $\left|\sin\left(\frac{1}{x}\right)\right| \leq 1 \ \ \forall x$, so we get

$$|f(x) - f(0)| = \left|x^2 \sin\left(\frac{1}{x}\right) - 0\right| \leq x^2$$

In addition we have that $|x - x_0| = |x| < \delta$. So, if we square $|x|$ then we get $|x|^2$ which we know is $< \epsilon$. Hence, let $\delta = \sqrt{\epsilon}$. We thus obtain

$$|f(x) - f(0)| \leq |x|^2 < \delta^2 = \epsilon.$$

This completes the proof.

Example 42: Show that $f(x) = x^2$ is pointwise continuous at $x_0 = a$ for $a \in \mathbb{R}$.

Let $\delta > 0, \epsilon > 0$ be given. We want to show that $|x - a| < \delta \implies |x^2 - a^2| < \epsilon$. To achieve this, we rewrite $|x^2 - a^2| = |x - a||x + a|$. Without loss of generality, let $|x - a| < 1$. This implies $|x + a| = |x - a + 2a| < 1 + 2|a|$. So, we have that

$$|x - a||x + a| < |x - a|(1 + 2|a|) = \delta \cdot (1 + 2|a|) = \epsilon.$$

We have two choices for $\delta : 1$ or $\frac{\epsilon}{1+2|a|}$. We want to choose whichever is smaller. So

$$\delta = \min\left\{1, \frac{\epsilon}{1 + 2|a|}\right\}$$

Theorem 4.2: Sequential Definition of Continuity

We say f is **pointwise continuous** at $x = x_0 \in \text{Dom}(f)$ if and only if for each sequence $\{x_n\} \subset \text{Dom}\,(f)$ converging to x_0, $\lim f(x_n) = f(x_0)$.

Example 43: Show that $f(x) = \begin{cases} \frac{1}{x}\sin\left(\frac{1}{x^2}\right), & x \neq 0 \\ 0, & x = 0 \end{cases}$ is discontinuous at $x = 0$.
Idea: We want to find a sequence x_n such that $x_n \to 0$ but $f(x_n)$ diverges. For simplicity, let us find such x_n for which $\sin\left(\frac{1}{x_n^2}\right) = 1$. For $x_n = \frac{1}{2\pi n + \frac{\pi}{2}}$, we have that $x_n \to 0$ but $\lim \frac{1}{x_n} \to \infty$. Therefore we have a discontinuity.

Theorem 4.3: Operations on Continuous Functions

Let f, g be continuous at x_0 and $c \in \mathbb{R}$ be some constant. We have that
(1) $|f|$ is continuous at x_0.
(2) cf is continuous at x_0.
(3) $f + g$ is continuous at x_0.
(4) $\frac{f}{g}$ is continuous at x_0 provided that $g(x_0) \neq 0$.
(5) $f(g(x)), g(f(x))$ are both continuous at x_0.

Properties of Continuous Functions
We look at two main properties of continuous functions

Theorem 4.4: Extreme Value Theorem

Let f be a continuous function on $[a, b]$. We have that f satisfies two properties.
(1) f is bounded.
(2) f attains its maximum/minimum on $[a, b]$ i.e. $\exists x_0, y_0 \in [a, b]$ such that $f(x_0) \leq f(x) \leq f(y_0)$.

The proof for this theorem stems from the Bolzano-Weierstrauss theorem and the properties of bounded sequences. For oscillating functions like $\sin x, \cos x$, we often have to show that we can find unique, converging subsequences because the main sequences diverge by oscillation.

Example 44: Show that $\sin(x)$ attains a maximum/minimum on $[0, 4\pi]$.

We obviously know that $\sin(x)$ is continuous. Let x_n be a sequence such that

$$x_n = \begin{cases} \frac{\pi}{2} + \frac{1}{n}, & n = 2k - 1 \\ \frac{5\pi}{2} + \frac{1}{n}, & n = 2k \end{cases}$$

Therefore we have that $x_n \to \frac{\pi}{2}, \frac{5\pi}{2}$ as $n \to \infty$ and $\lim f(x_n) = 1$. We have that $\sin(x) \leq 1$ for all $x \in [0, 4\pi]$. A similar case for the minimum can be applied for $x = \frac{3\pi}{2}, \frac{7\pi}{2}$.

Theorem 4.5: Intermediate Value Theorem (IVT)

Suppose f is continuous on an interval (a, b), then there exists a $c \in (a, b)$ such that for $f(a) < y < f(b)$, $f(c) = y$.

An incredibly useful result from the IVT is that we can use this to show the existence of roots of continuous functions. In fact, we can translate a lot of problems into root problems. Let the following example demonstrate this.

Example 45: Let f, g be two continuous functions on $[a, b]$ such that $f(a) \geq g(a)$ and $f(b) \leq g(b)$. Prove $f(x_0) = g(x_0)$ for at least one $x_0 \in [a, b]$.

Idea: Let us define a new function $h(x) = f(x) - g(x)$. At $h(a), f(a) \geq g(a)$ implies $h(a) > 0$, and $h(b) < 0$ by the same logic. Therefore, the IVT guarantees that $\exists x_0 \in [a, b]$ such that $h(x_0) = 0$. This implies $f(x_0) - g(x_0) = 0$ and $f(x_0) = g(x_0)$ for at least x_0, as desired.

Uniform Continuity

Definition 4.6: Uniform Continuity

Let a function f and interval I be given. We say f is **uniformly continuous** in I if for each $\epsilon > 0, \exists \delta > 0$ such that $|f(x) - f(y)| < \epsilon$ for all $x, y \in I$ with $|x - y| < \delta$.
Remark: Recall that for pointwise continuity, δ is dependent on the point x_0 and ϵ. For uniform continuity, δ is not dependent on x, y, only ϵ!

Example 46: Show $f(x) = \frac{1}{x^2}$ is uniformly continuous on $[1, \infty)$.

Let $\epsilon > 0, \delta > 0$ be given. We want to find δ such that

$$\left| \frac{1}{x^2} - \frac{1}{y^2} \right| < \epsilon$$

We have that

$$\left| \frac{1}{x^2} - \frac{1}{y^2} \right| = \left| \frac{y^2 - x^2}{x^2 y^2} \right| = \left| \frac{x + y}{x^2 y^2} \right| |x - y| = \left| \frac{1}{x^2 y} + \frac{1}{x y^2} \right| |x - y|$$

And so

$$\left|\frac{1}{x^2y}+\frac{1}{xy^2}\right||x-y| \leq 2|x-y| < \epsilon = 2\delta$$

Setting $\delta = \frac{\epsilon}{2}$ completes the proof.

Theorem 4.7: Continuity implies Uniform Continuity on a Closed Interval

If f is continuous on $[a, b]$, then f is uniformly continuous on $[a, b]$.

Note that the converse is generally not true. Uniform continuity is a tighter case than continuity.

Example 47: Show that $f(x) = x^3$ is uniformly continuous on $[0, 1]$, but not uniformly continuous on $\mathbb{R}$.

(1) As for the first case, we simply apply the above Theorem. $f(x)$ is clearly continuous on $[0, 1]$, so its uniform continuity comes for free.

(2) The second case isn't as obvious. We need to use the sequential definition of continuity, so let $(x_n), (y_n)$ be two sequences such that $(x_n), (y_n) \to 0$. This time, we want to show $|f(x_n) - f(y_n)| \geq \epsilon$ for some $\epsilon > 0$ Let $x_n = n + \frac{1}{n}, y_n = n$. So $|f(x_n - f(y_n)| = \left|\left(n + \frac{1}{n}\right) - n^3\right| = 3n$. We know that $3n \geq 3$ for all $n \in \mathbb{N}$, so choosing $\epsilon = 3$ completes the proof.

Theorem 4.8: Uniform Continuity and Cauchy Sequences

If f is uniformly continuous on I and $\{s_n\}$ is a sequence in I, then $f(\{s_n\})$ is Cauchy.

Example 48: Show that $f(x) = \frac{1}{x^2}$ is not uniformly continuous on $(0, 1)$.

Here we let $s_n = \frac{1}{n}$, and so $f(s_n) = n^2$. Because s_n is a convergent sequence, it is also Cauchy. However, $f(s_n)$ is not Cauchy, so clearly $f(x)$ is not uniformly continuous on $(0, 1)$.

5 Acknowledgements

Some definitions, example problems, and exercises were adapted from Neil Donaldson's Math 140A Notes and *Elementary Analysis: The Theory of Calculus, Second Edition* by Kenneth A. Ross
See https://www.math.uci.edu/ ndonalds/math140a/math140a.html