Math 112B Notes: Ryan Gomberg

1 Fourier Analysis in One Variable

Parseval's Identity

$$\frac{1}{\pi}\int_{-\pi}^{\pi}(f(x))^2dx = \frac{a_0^2}{2} + \sum_{N=1}^{\infty}(a_n^2 + b_n^2)$$

Generalized Parseval's Identity
Let $f(x), f^*(x) \in L^2([-\pi, \pi])$. Then,

$$\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)f^*(x)dx = \frac{a_0 a_0^*}{2} + \sum_{N=1}^{\infty}(a_n a_n^* + b_n b_n^*)$$

Error Bound
The error for a Fourier Series approximating $f(x)$ can be computed by

$$|f(x) - s_N(x)| = \left(\frac{1}{\pi}\int_{-\pi}^{\pi}(f'(x))^2dx - \sum_{n=1}^{N} n^2(a_n^2 + b_n^2)\right)^{\frac{1}{2}} \left(\frac{\pi^2}{6} - \sum_{n=1}^{N}\frac{1}{n^2}\right)^{\frac{1}{2}}$$

...where $s_N(x)$ is the partial sum of the Fourier Series for f.

Pointwise Convergence of Fourier Series

(1) If $\int_{-\pi}^{\pi}|\frac{F(x_0+\tau)-F(x_0)}{\tau}|d\tau$ is finite, then we have pointwise convergence at x_0 (Dini's Test)

(2) Suppose f is bounded, has finite extrema, and has finite discontinuities. Then, the series converges to x_0 at $\frac{f(x_0^-)+f(x_0^+)}{2}$ (Dirichlet's Theorem)

Uniform Convergence of Fourier Series Let $f(x)$ be a continuous 2π-periodic function that satisfies the below conditions:

(1) $f'(x)$ is continuous, except for a finite amount of points
(2) $\int_{-\pi}^{\pi}(f'(x))^2dx$ is finite
(3) $f(x) - f(-\pi) = \int_{-\pi}^{x} f'(t)dt$ for all x

...then the Fourier Series of f converges uniformly to $f(x)$.
To show convergence for all x, prove that this holds for $f(x - 2\pi m)$

Convergence in the Mean
Uniform convergence implies convergence in the mean. $f(x) \in L^2([-\pi, \pi])$ means f converges in the l_2 norm.

Formal definition: A sequence of functions $\{f_k(x)\}_{k\geq 1}$ defined over $a \leq x \leq b$ is said to converge in the mean to $g(x)$ if

$$\lim_{k\to\infty}\int_a^b (f_k(x) - g(x))^2 dx = 0$$

Completeness

We say a set of orthogonal functions is complete if the Fourier Series based off the set of functions converges in the mean.

Change of Scale

Let $f(x)$ be a function defined on $a \leq x < b$. We can apply a *change of scale* such that our new domain $\overline{x}$ satisfies $-\pi < \overline{x} < \pi$. So, let

$$\overline{x} = 2\pi\left(\frac{x - \frac{1}{2}(a+b)}{b-a}\right) \Longrightarrow x = \frac{b-a}{2\pi}\overline{x} + \frac{1}{2}(a+b)$$

Thus, $F(\overline{x})$ has the Fourier Series

$$F(\overline{x}) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\cos(n\overline{x}) + b_n\sin(n\overline{x}))$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos\left(2n\pi\left(\frac{x - \frac{1}{2}(a+b)}{b-a}\right)\right) + b_n\sin\left(2n\pi\left(\frac{x - \frac{1}{2}(a+b)}{b-a}\right)\right)\right)$$

The sine and cosine series is as such

$$a_0 = \frac{2}{b-a}\int_a^b f(x)dx$$

$$a_n = \frac{2}{b-a}\int_a^b f(x)\cos\left(\frac{\pi n}{b-a}(x-a)\right)dx$$
$$b_n = \frac{2}{b-a}\int_a^b f(x)\sin\left(\frac{\pi n}{b-a}(x-a)\right)dx$$

Example: Let $f(x) = e^x, 1 \leq x < 2$.
$\overline{x} = 2\pi\left(x - \frac{1}{2}(1+2)\right) = \pi(2x-3)$
So, its Fourier Series is

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\cos(n\pi(2x-3)) + b_n\sin(n\pi(2x-3)))$$

The sine and cosine series is as such

$$a_0 = 2\int_1^2 e^x dx = 2(e^2 - e)$$

$$a_n = 2\int_1^2 f(x)\cos(\pi n(x-1))dx$$

$$b_n = 2\int_1^2 f(x)\sin(\pi n(x-1))dx$$

Pointwise Convergence Example: Let $f(x) = e^x, 1 \leq x < 2$.
We have that $f(x)$ is bounded from $e \leq x < e^2$ and contains finitely many extrema and discontinuities within (1, 2). So, $\forall x$, the Fourier Series for f converges to $\frac{f(x^-)+f(x^+)}{2}$.

Uniform Convergence Example: $f(x) = e^x, 1 \leq x < 2$
First, we show that $\int_a^b (f'(x))^2 dx < \infty$.

$$\int_1^2 e^{2x} dx = \frac{1}{2}(e^4 - e^2) < \infty$$

Next, $f(x) - f(1) = \int_1^x (e^t)' dt$. Both sides are equivalent to $(e^x - e)$

Now, consider an extension of f with periodicity $= (b-a)m = 1m$, where $m \in \mathbb{N}$. Then,

$$f(x) = e^{x+m} = f(1) + \int_1^{x+m} e^t dt = f(1) + \int_{-m}^{x} e^{t-m} dt$$

$$= f(1) + \int_1^2 e^t dt + \int_2^3 e^{t-1} dt + \int_3^4 e^{t-2} dt + ... + \int_0^m e^t dt$$

$$= e + (e^2 - e) + (e^2 - e) + (e^2 - e) + ... + (e^2 - e) = e + m(e^2 - e) \neq \int_{-m}^{x} e^{t-m} dt$$

.
Therefore, f does not converge uniformly.

Convergence in the Mean Example: Let $f(x) = e^x, 1 \leq x < 2$.
We show $\int_a^b (f(x))^2 dx < \infty$

$$\int_1^2 e^{2x} dx = \frac{1}{2}(e^4 - e^2) < \infty$$

We have shown that f converges in the mean.

2 Homogeneous PDEs in Two Variables

Laplace's Equation in a Rectangle
We consider the setup

$$\begin{cases} \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 & 0 < x < \pi, 0 < y < A \\ u(0, y) = u(\pi, y) = 0 & 0 < y < A \\ u(x, A) = 0 \\ u(x, 0) = f(x) \end{cases}$$

Apply the standard separation of variables, guessing $u(x, y) = X(x)Y(y)$. After solving each ODE we obtain:

$$X_n(x) = \sin(nx), Y_n(y) = \sinh(n(A - y))$$

To find $Y_n(y)$, it involves solving for a constant c_1 (or c_2) with respect to the other constant.
Hence, the solution $u(x, y)$ is the following series

$$u(x, y) = \sum_{n=1}^{\infty} b_n \frac{\sinh(n(A - y))}{\sinh(nA)} \sin(nx)$$

$\frac{1}{\sinh(nA)}$ is obtained from the boundary condition $u(x, 0) = f(x)$.

Laplace's Equation in a Circle
The PDE is still of the form $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$, but now we have a new boundary condition: $x^2 + y^2 < 1$. Our new PDE becomes:

$$\begin{cases} \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 & x^2 + y^2 < 1 \\ u = f(\theta), 0 < \theta \leq 2\pi \end{cases}$$

We apply separation of variables: guess $u(r, \theta) = R(r)\Theta(\theta)$. Solving for Θ is a familiar situation to previous PDEs, as we will obtain $\Theta_n(\theta) = A_n \sin(n\theta) + B_n \cos(n\theta)$.

To solve $R(r)$, we will get a characteristic ODE $r^2 R'' + rR' - n^2 R = 0$. However, we also have to analyze two separate cases: $n = 0$ and $n \geq 1$.

If $n = 0$, then we have $R(r) = a + \ln |r| + b$.

If $n \geq 1$, the ODE is a characteristic polynomial. Therefore, we guess $R(r) = r^\alpha$, and the final solution is $n = \pm\alpha$. So, $R(r) = r^n - r^{-n}$.
However, we need the series to converge. Because $r < 1$ by assumption, r^{-n} diverges and so we exclude r^{-n} from our final solution.

The series solution to our PDE is

$$u(r,\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} r^n (A_n \sin(n\theta) + B_n \cos(n\theta)).$$

with A_n and B_n complying to the standard Fourier Coefficients

$$A_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\cos(n\theta)d\theta, B_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\sin(n\theta)d\theta$$

.

Suppose the circle had a radius r as opposed to 1 such that boundary condition is $x^2 + y^2 < a$. Then, we can apply a *change of scale*. Let $r = \sqrt{a}\overline{r}$. Consider a new solution $v(\overline{r},\theta) = u(\sqrt{a}\overline{r},\theta)$. We now have that $\triangle v$ satisfies our standard boundary condition $\overline{r} < 1$. Solve the PDE as you would in the above case, and then rewrite $\overline{r}$ in terms of r in your final solution.

Other boundary types: Annulus, Wedge

On an annulus, we have a different condition

$$\begin{cases} \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 & a < r < b, \theta_1 \le \theta < \theta_2 \\ u(a,\theta) = f_1(\theta), u(b,\theta) = f_2(\theta) \end{cases}$$

The procedure is mostly similar. Apply separation of variables and obtain the series solution. However, we have to be cautious of how we set it up. For example, if $u(a,\theta) = 0$, we have to make sure $f(b,\theta) = a_n \cos(n\theta) + b_n \sin(n\theta)$.

On a wedge, we have the boundary conditions

$$\begin{cases} \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 & 0 < r \le a, 0 \le \theta < \theta \\ u(r,0) = f(r), u(a,\theta) = f(\theta) \end{cases}$$

Poisson's Integral

ASSUME that $u(r,\theta)$ is a solution to Laplace's Equation on a circle with radius R. Then,

$$u(r,\theta) = \frac{R^2 - r^2}{2\pi}\int_{-\pi}^{\pi} \frac{f(\phi)}{r^2 + R^2 - 2Rr\cos(\theta - \phi)} d\phi$$

.

3 Nonhomogeneous Problems and Green's Function

Green's Function for Initial Value Problems
Let an ODE be of the form

$$u''(x) + p(x)u'(x) + q(x)u(x) = f(x), \quad u(a) = A, u'(a) = B$$

We can first guess **two linearly independent solutions** $v_1(x), v_2(x)$ of the non-homogeneous equation

$$v''(x) + p(x)v'(x) + q(x)v(x) = 0$$

Now, let's introduce the Wronskian, which outputs the determinant of a matrix containing v_1, v_2. More specifically,

$$W(v_1, v_2) = v_1 v_2' - v_1' v_2$$

Finally, we denote the one-sided Green's Function

$$R(x, \xi) = \frac{-v_1(x)v_2(\xi) + v_1(\xi)v_2(x)}{W(v_1, v_2)}$$

The solution to such ODE is

$$u(x) = \int_\alpha^x R(x, \xi) f(\xi) d\xi$$

Remark: Note that the above solution is final **IF** the initial values are zero. If that is not the case, we must add $c_1 v_1(x) + c_2 v_2(x)$ to our solution, using our initial conditions to find such c_1, c_2.

Green's Function for Boundary Value Problems
Let an ODE be of the form

$$u''(x) + p(x)u'(x) + q(x)u(x) = f(x), \quad u(a) = A, u(b) = B$$

Then

$$G(x, \xi) = \begin{cases} G^-(x, \xi) & \xi \leq x \\ G^+(x, \xi) & \xi \geq x \end{cases}$$

and

$$u(x) = \int_a^b G(x, \xi) f(\xi) d\xi - A\frac{\partial G}{\partial \xi}(x, a) + B\frac{\partial G}{\partial \xi}(x, b)$$

$$= \int_a^x G^+(x, \xi) f(\xi) d\xi + \int_x^b G^-(x, \xi) f(\xi) d\xi - A\frac{\partial G}{\partial \xi}(x, a) + B\frac{\partial G}{\partial \xi}(x, b)$$

Here

$$G^{-}(x,\xi) = \frac{[v_1(x)v_2(b) - v_1(b)v_2(x)][v_1(\xi)v_2(a) - v_1(a)v_2(\xi)]}{[v_1(a)v_2(b) - v_1(b)v_2(a)][v_1(\xi)v_2'(\xi) - v_1'(\xi)v_2(\xi)]}$$

and

$$G^{+}(x,\xi) = \frac{[v_1(x)v_2(a) - v_1(a)v_2(x)][v_1(\xi)v_2(b) - v_1(b)v_2(\xi)]}{[v_1(a)v_2(b) - v_1(b)v_2(a)][v_1(\xi)v_2'(\xi) - v_1'(\xi)v_2(\xi)]}.$$

Alternatively, you can write expressions for $G(a,\xi), G(b,\xi)$ and set them equal to our boundary values.

Nonhomogeneous Heat Equation

Say we have a PDE of the following form

$$\begin{cases} \frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = F(x,t) & 0 < x < \pi, t > 0. \\ u(0,t) = u(\pi,t) = u(x,0) = 0 & 0 \le x \le \pi, t > 0. \end{cases}$$

If a solution $u(x,t)$ exists, then we can represent $u(x,t)$ as a Fourier Series

$$u(x,t) \sim \sum_{n=1}^{\infty} b_n(t)\sin(nx).$$

We can apply a Fourier Sine Transform because $b_n(t)$ uniquely describes the solution $u(x,t)$. To find such $b_n(t)$, we first want to find the standard Fourier coefficients written solely as a function of t. We will call these set of coefficients $B_n(t)$. So

$$B_n(t) = \frac{2}{\pi}\int_0^{\pi} F(x,t)\sin(nx)dx.$$

Now to solve $b_n'(t) + n^2 b_n(t) = B_n(t)$ you apply the integrating factor and so

$$b_n(t) = \int_0^t e^{-n^2(t-\tau)}B_n(\tau)d\tau = e^{-n^2 t}\int_0^t e^{n^2\tau}B_n(\tau)d\tau$$

And so

$$u(x,t) = \sum_{n=1}^{\infty} b_n(t)\sin(nx)$$

Example

$$\begin{cases} \frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = x(\pi - x)\sin t & 0 < x < \pi, t > 0. \\ u(x,0) = u(0,t) = u(\pi,t) = 0 & 0 \le x \le \pi, t \ge 0. \end{cases}$$

First, find the sine series for $f(x) = x(\pi - x)$

$$B_n = \frac{2}{\pi}\int_0^{\pi}(\pi x - x^2)\sin(nx)dx$$

The terms vanish when n is even so we get

$$B_{2k-1}(t) = \frac{8}{\pi(2k-1)^3} \sin t$$

To find our $b_{2k-1}(t)$ terms we apply the above integral

$$b_{2k-1} = \int_0^t e^{-(2k-1)^2(t-\tau)} \left(\frac{8}{(2k-1)^3 \pi} \right) \sin(\tau) d\tau$$

After integrating by parts twice, the coefficients for b_{2k-1} is

$$b_{2k-1}(t) = \frac{8((2k-1)^2 \sin t - \cos t + e^{-(2k-1)t})}{(\pi(2k-1)^3)((2k-1)^4+1)}$$

Hence our solution $u(x,t)$ is

$$u(x,t) = \sum_{k=1}^{\infty} \frac{8((2k-1)^2 \sin t - \cos t + e^{-(2k-1)t})}{(\pi(2k-1)^3)((2k-1)^4+1)} \sin((2k-1)x)$$

Nonhomogenous Laplace's Equation on a Rectangle
Consider the PDE

$$\begin{cases} \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = F(x,y) & 0 < x < \pi, 0 < y < A. \\ u(0,y) = u(\pi,y) = u(x,0) = u(x,A) = 0 & 0 \leq x \leq \pi, 0 \leq y \leq A. \end{cases}$$

Compared to the Nonhomogeneous Heat Equation, the general approach is exactly the same. Find the coefficients $B_n(y)$ and then solve for $b_n(y)$. Find the Fourier Coefficients for $B_n(y) = \frac{2}{\pi} \int_0^\pi F(x,y) \sin(nx) dx$. Now we solve the nonhomogeneous ODE for $b_n(y)$.

$$b_n''(y) - n^2 b_n(y) = B_n(y)$$

This can be solved using the Green's Function, but generally an easier approach will be guessing the particular solution $b_{n,p}(y)$, which depends on the form of $B_n(y)$.
As for the homogeneous solution of the ODE, the solution for each $b_n(y)$ is

$$b_{n,h}(t) = c_1 e^{ny} + c_2 e^{-ny}$$

So the general solution of the ODE, for each n, is

$$b_n(y) = b_{n,p}(y) + b_{n,h}(y) = b_{n,p}(y) + c_1 e^{ny} + c_2 e^{-ny}.$$

Use the boundary conditions $u(x,0)$ and $u(x,A)$ to solve for coefficients c_1, c_2.

The final solution of the PDE is

$$u(x,y) = \sum_{n=1}^{\infty} b_n(y)\sin(nx)$$

Example

$$\begin{cases} \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = y(1-y)\sin^3(x) & 0 < x < \pi, 0 < y < 1. \\ u(0,y) = u(\pi,y) = u(x,0) = u(x,1) = 0 & 0 \le x \le \pi, 0 \le y \le 1. \end{cases}$$

While we can find the Fourier Series for $B_n(y)$ by integrating, we know that $\sin^3(x) = \frac{3}{4}\sin x - \frac{1}{4}\sin(3x)$. Therefore we will have to solve two ODEs, one for $b_1(y)$ and $b_3(y)$.
We have that $B_1(y) = \frac{3}{4}y(1-y)$. We can ignore $\sin x$ because we are solving the ODE with respect to y. In order to solve $b_1''(y) - b_1(y) = \frac{3}{4}y(1-y)$, we need to guess the particular solution. Here we can guess $b_{1,p} = ay^2 + by + c$, where $b_{1,p}'' = 2a$. Our system becomes

$$2a - ay^2 - by - c = \frac{3}{4}y - \frac{3}{4}y^2.$$

Solving the system gives us $a = \frac{3}{4}, b = -\frac{3}{4}, c = -\frac{3}{2}$ and so

$$b_{1,p}(y) = \frac{3}{4}y^2 - \frac{3}{4}y - \frac{3}{2}.$$

Now, for the homogeneous solution $b_{1,h}$, we have

$$b_{1,h} = c_1 e^y + c_2 e^{-y}$$

Hence our final $b_1(y)$ is

$$b_1(y) = \frac{3}{4}y^2 - \frac{3}{4}y + \frac{3}{2} + c_1 e^y + c_2 e^{-y}.$$

Using our boundary conditions $b_1(0) = 0, b_1(1) = 0$, we can solve for c_1 and c_2 and obtain the solution to this ODE

$$b_1(y) = \frac{3}{4}\left(y^2 - y + 2 - 2\left(1 + \frac{e^2-1}{e^2+e}\right)e^y + 2\left(\frac{e^2-1}{e^2+e}\right)e^{-y}\right)$$

Solving for $b_3(y)$ follows the same procedure. Instead, we solve the ODE $b_3''(y) - 9b_3(y) = -\frac{1}{4}y(1-y)$. We apply the same guess for our characteristic solution and our homogeneous solution is $b_{3,h} = c_1 e^{3y} + c_2 e^{-3y}$. The solution $b_3(y)$ for this specific ODE is

$$b_3(y) = \left(-\frac{1}{36}y^2 + \frac{1}{36}y + \frac{1}{162} - \frac{1}{162}\left(\frac{e^3-e^6}{e^6-1} + 1\right)e^{3y} + \frac{1}{162}\left(\frac{e^3-e^6}{e^6-1}\right)e^{-3y}\right)$$

Hence the solution $u(x,y)$ is

$$u(x,y) = \sum_{n=1}^{\infty} b_n(y)\sin(nx) = b_1(y)\sin x + b_3(y)\sin(3x)$$

$$= \frac{3}{4}\left(y^2 - y + 2 - 2\left(1 + \frac{e^2-1}{e^2+e}\right)e^y + 2\left(\frac{e^2-1}{e^2+e}\right)e^{-y}\right)\sin x$$

$$+\left(-\frac{1}{36}y^2 + \frac{1}{36}y + \frac{1}{162} - \frac{1}{162}\left(\frac{e^3-e^6}{e^6-1}+1\right)e^{3y} + \frac{1}{162}\left(\frac{e^3-e^6}{e^6-1}\right)e^{-3y}\right)\sin(3x)$$

Nonhomogeneous Laplace's Equation on a Disk
Consider the following PDE with corresponding boundary conditions

$$\begin{cases} \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = F(r,\theta) & r < R, -\pi \leq \theta < \pi \\ u(R,\theta) = 0 & -\pi \leq \theta < \pi \end{cases}$$

Much like the other nonhomogeneous problems, we seek coefficients $a_n(r), b_n(r)$ that solve each ODE and conditions. Here we find $a_n(r), b_n(r)$ such that

$$a_n''(r) + \frac{1}{r}a_n'(r) - \frac{n^2}{r^2}a_n(r) = A_n(r)$$

$$b_n''(r) + \frac{1}{r}b_n'(r) - \frac{n^2}{r^2}b_n(r) = B_n(r)$$

where

$$A_n(r) = \frac{1}{\pi}\int_{-\pi}^{\pi} F(r,\theta)\cos(n\theta)d\theta, B_n(r) = \frac{1}{\pi}\int_{-\pi}^{\pi} F(r,\theta)\sin(n\theta)d\theta$$

In the case where $n = 0$, we have that $a_n = a_n''(r) + \frac{1}{r}a_n'(r) = 0$. Multiplying by r^2 yields

$$r^2 a_n''(r) + r a_n'(r) = 0$$

We can solve the ODE using the integrating factor, giving the result

$$a_0(r) = \int_0^r \ln\left(\frac{r}{\rho}\right) A_0(\rho)\rho d\rho + C$$

The integrand $\rho d\rho$ comes from rewriting the integral in polar form. With the initial condition $a_0(R) = 0$ gives us

$$C = -\int_0^R \ln\left(\frac{R}{\rho}\right) A_0(\rho)\rho d\rho$$

So

$$a_0(r) = \int_0^r \ln\left(\frac{r}{R}\right) A_0(\rho)\rho d\rho + \int_r^R \ln\left(\frac{\rho}{R}\right) A_0(\rho)\rho d\rho$$

For $n \geq 1$, we can use Green's Function to find the solution. Guess $v_1(r) = r^n, v_2(r) = r^{-n}$. Simple computation yields the coefficients

$$a_n(r) = \frac{1}{2n}\int_0^r \left[\left(\frac{r}{R}\right)^n - \left(\frac{R}{r}\right)^n\right]\left(\frac{\rho}{R}\right)^n A_n(\rho)\rho d\rho + \frac{1}{2n}\int_r^R \left[\left(\frac{\rho}{R}\right)^n - \left(\frac{R}{\rho}\right)^n\right]\left(\frac{r}{R}\right)^n A_n(\rho)\rho d\rho$$

$$b_n(r) = \frac{1}{2n}\int_0^r \left[\left(\frac{r}{R}\right)^n - \left(\frac{R}{r}\right)^n\right]\left(\frac{\rho}{R}\right)^n B_n(\rho)\rho d\rho + \frac{1}{2n}\int_r^R \left[\left(\frac{\rho}{R}\right)^n - \left(\frac{R}{\rho}\right)^n\right]\left(\frac{r}{R}\right)^n B_n(\rho)\rho d\rho$$

Green's Function for Nonhomogeneous PDEs

Recall Laplace's Equation on a Rectangle

$$\begin{cases} \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = F(x,y) & 0 < x < \pi, 0 < y < A. \\ u(0,y) = u(\pi,y) = u(x,0) = u(x,A) = 0 & 0 \leq x \leq \pi, 0 \leq y \leq A. \end{cases}$$

The corresponding Green's Function is as such

$$u(x,y) = \int_0^A \int_0^\pi G(x,y,\xi,\eta)F(\xi,\eta)d\xi d\eta$$

where

$$G(x,y,\xi,\eta) = \frac{2}{\pi}\sum_{n=1}^{\infty} G_n(y,\eta)\sin(nx)\sin(n\eta)$$

and

$$G_n(y,\eta) = \begin{cases} \frac{\sinh(n(\eta - A))\sinh(ny)}{n\sinh(nA)}, & 0 \leq y \leq \eta \\ \frac{\sinh(n\eta))\sinh(n(y-A))}{n\sinh(nA)}, & \eta \leq y \leq A \end{cases}$$

Now, recall Laplace's Equation on a Circle

$$\begin{cases} \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = F(r,\theta) & r < R, -\pi \leq \theta < \pi \\ u(R,\theta) = 0 & -\pi \leq \theta < \pi \end{cases}$$

The idea is that we want to find the corresponding Green's Function $G(r,\theta;\rho,\phi)$. We have that

$$u(r,\theta) = \int_0^R \int_{-\pi}^{\pi} G(r,\theta;\rho,\phi)F(\rho,\phi)\rho d\rho d\phi$$

where

$$G(r,\theta;\rho,\phi) = \frac{1}{4\pi}\left\{-\ln\left[R^2 + \frac{r^2\rho^2}{R^2} - 2r\rho\cos(\theta-\phi)\right] + \ln[r^2+\rho^2 - 2r\rho\cos(\theta-\phi)]\right\}$$

Note that the derivation comes from deriving the complex form of the Green's Function. In addition, the arguments inside ln resemble the Law of Cosines formula.

4 Fourier Series in Two Variables

Double Fourier Series

Consider a 2π-periodic function $f(x,y)$ in **both** x and y. Then its Fourier Series can be derived as

$$f(x,y) \sim \frac{1}{4}a_{00} + \frac{1}{2}\sum_{m=1}^{\infty}[a_{0m}\cos(my) + b_{0m}\sin(my)] + \sum_{n=1}^{\infty}[a_{n0}\cos(nx) + c_{n0}\sin(nx)]$$

$$+\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}[a_{nm}\cos(nx)\cos(my)+b_{nm}\cos(nx)\sin(my)+c_{nm}\sin(nx)\cos(my)+d_{nm}\sin(nx)\sin(my)]$$

with corresponding coefficients

$$a_{00} = \frac{1}{\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(x,y)dxdy$$

$$a_{0m} = \frac{1}{\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(x,y)\cos(my)dxdy$$

$$b_{0m} = \frac{1}{\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(x,y)\sin(my)dxdy$$

$$a_{n0} = \frac{1}{\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(x,y)\cos(nx)dxdy$$

$$c_{n0} = \frac{1}{\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(x,y)\sin(nx)dxdy$$

$$a_{nm} = \frac{1}{\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(x,y)\cos(nx)\cos(my)dxdy$$

$$b_{nm} = \frac{1}{\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(x,y)\cos(nx)\sin(my)dxdy$$

$$c_{nm} = \frac{1}{\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(x,y)\sin(nx)\cos(my)dxdy$$

$$d_{nm} = \frac{1}{\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(x,y)\sin(nx)\sin(my)dxdy$$

Parseval's Identity for Double Fourier Series

$$\frac{1}{\pi^2}\int_{-\pi}^{\pi} f^2(x,y)dxdy = \frac{1}{4}a_{00}^2+\frac{1}{2}\sum_{m=1}^{\infty}(a_{0m}^2+b_{0m}^2)+\frac{1}{2}\sum_{n=1}^{\infty}(a_{n0}^2+c_{n0}^2)+\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}(a_{nm}^2+b_{nm}^2+c_{nm}^2+d_{nm}^2)$$

Example: Find the double sine series of $f(x,y) = x^2y^2$. Also discuss its uniform convergence.
We only have to find the coefficient d_{nm}

$$d_{nm} = \frac{4}{\pi^2}\int_0^\pi\int_0^\pi x^2y^2\sin(nx)\sin(my) = \left[\frac{\pi^2(-1)^n}{n} + \frac{2((-1)^n-1)}{n^3}\right]\left[\frac{\pi^2(-1)^m}{m} + \frac{2((-1)^m-1)}{m^3}\right]$$

So

$$f(x,y) \sim \frac{4}{\pi^2}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left[\frac{\pi^2(-1)^n}{n} + \frac{2((-1)^n-1)}{n^3}\right]\left[\frac{\pi^2(-1)^m}{m} + \frac{2((-1)^m-1)}{m^3}\right]\sin(nx)\sin(my)$$

We do not have uniform convergence as the double series does not have absolute convergence.

Example: Find the full Fourier Series for $f(x,y) = \sin^2(x)y^3, -\pi < x, y < \pi$
To simplify a lot of the computation, the only nonzero coefficients are b_{0m} and b_{nm}. The other coefficients vanish because the integrands are odd functions. Now we compute b_{0m}.

$$b_{0m} = \frac{1}{\pi^2}\int_{-\pi}^{\pi}\sin^2 x\sin(my)y^3dxdy = 4\pi\left(\frac{6\pi(-1)^m - \pi^3(-1)^m}{m}\right)$$

While computing b_{nm}, we evaluate the same integral for y in b_{0m}. When we integrate

$$\int_{-\pi}^{\pi}\sin^2(x)\sin(nx)dx$$

we notice that this integral is nonzero only when $n = 2$. When $n = 2$, the integral comes out to $\frac{\pi}{2}$. So we can rename the coefficient to b_{2m} and so

$$b_{2m} = \frac{6(-1)^m - \pi^2(-1)^m}{m}$$

Therefore, the Fourier Series expansion is

$$f(x,y) \sim 2\pi\sum_{m=1}^{\infty}\frac{6\pi(-1)^m - \pi^3(-1)^m}{m}\sin(my) + \sum_{m=1}^{\infty}\frac{6(-1)^m - \pi^2(-1)^m}{m}\cos(2x)\cos(my)$$

5 Homogeneous PDEs in Three Variables

The Heat Equation in a Square
We consider

$$\begin{cases} \frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = 0, & 0 < x, y < \pi, t > 0 \\ u(0, y, t) = u(\pi, y, t) = u(x, 0, t) = u(x, \pi, t) = 0 & \\ u(x, y, 0) = g(x, y), & 0 \leq x, y \leq \pi \end{cases}$$

We apply the standard separation of variables, this time with 3 variables: X, Y, T. So guess a solution of the form $u(x, y, t) = X(x)Y(y)T(t)$. Dividing by XYT yields

$$\frac{T'}{T} - \frac{X''}{X} - \frac{Y''}{Y} = 0$$

We first solve the case for $X(x)$. We consider

$$\frac{T'}{T} - \frac{Y''}{Y} = \frac{X''}{X}$$

But we know that both sides must be constant, so let

$$\frac{T'}{T} - \frac{Y''}{Y} = \frac{X''}{X} = C_1.$$

We solve the case for $X(x)$.

$$\begin{cases} X'' - C_1 X = 0 & 0 < x < \pi \\ X(0) = X(\pi) = 0 \end{cases}$$

We already know this yields the solution $X(x) = \sin(nx)$, where $C_1 = -n^2$ for $n \geq 1$. For solving $Y(y)$ and $T(t)$, we have the setup

$$\frac{T'}{T} = \frac{Y''}{Y} + C_1$$

Again, both sides are equal to a constant, say C_2. So

$$\frac{T'}{T} = \frac{Y''}{Y} + C_1 = C_2$$

We solve $Y(y)$

$$\begin{cases} Y'' + (C_1 - C_2)Y = 0 & 0 < y < \pi \\ Y(0) = Y(\pi) = 0 \end{cases}$$

This will also yield $Y(y) = \sin(my)$, this time $C_1 - C_2 = m^2$ for $m \geq 1$.

Since $C_1 - C_2 = m^2$, this gives $C_2 = C_1 - m^2 = m^2 + n^2$. As for solving $T(t)$, we have the equation

$$T' + (m^2 + n^2)T = 0 \Longrightarrow e^{-(m^2+n^2)t}.$$

By verifying the boundary conditions and checking for convergence, we conclude that the double series solution

$$u(x, y, t) = \sum_{n=1}^{\infty}\sum_{n=1}^{\infty} d_{nm} e^{-(m^2+n^2)t} \sin(nx)\sin(my)$$

satisfies the Heat Equation.

Example: Solve the Heat Equation in a Square

$$\begin{cases} \frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = 0, & 0 < x, y < \pi, t > 0 \\ u(0, y, t) = u(\pi, y, t) = u(x, 0, t) = u(x, \pi, t) = 0 \\ u(x, y, 0) = \sin^3 x \sin^5 y, & 0 \leq x, y \leq \pi \end{cases}$$

We have that $\sin^3 x = \frac{3}{4}\sin x - \frac{1}{4}\sin(3x)$ and $\sin^5 y = \frac{5}{8}\sin y - \frac{5}{16}\sin(3y) + \frac{1}{16}\sin(5y)$ (See Appendix).
By expansion

$$\begin{aligned} \sin^3 x \sin^5 y = \frac{15}{32}\sin x \sin y - \frac{15}{64}\sin x \sin(3y) + \frac{3}{64}\sin x \sin(5y) \\ - \frac{5}{32}\sin(3x)\sin y + \frac{5}{64}\sin(3x)\sin(3y) - \frac{1}{64}\sin(3x)\sin(5y) \end{aligned}$$

So $d_{1,1} = \frac{15}{32}, d_{1,3} = -\frac{15}{64}, d_{1,5} = \frac{3}{64}, d_{3,1} = -\frac{5}{32}, d_{3,3} = \frac{5}{64}, d_{3,5} = -\frac{1}{64}$.
Our final solution is

$$\begin{aligned} u(x, y, t) = \frac{15}{32}e^{-2t}\sin x \sin y - \frac{15}{64}e^{-10t}\sin(x)\sin(3y) + \frac{3}{64}e^{-26t}\sin x \sin(5y) \\ - \frac{5}{32}e^{-10t}\sin(3x)\sin y + \frac{5}{64}e^{-18t}\sin(3x)\sin(3y) - \frac{1}{64}e^{-34t}\sin(3x)\sin(5y) \end{aligned}$$

Laplace's Equation in a Cube
Consider the PDE

$$\begin{cases} \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} - \frac{\partial^2 u}{\partial z^2} = 0, & 0 < x, y, z < \pi \\ u(0, y, z) = u(\pi, y, z) = u(x, 0, z) = u(x, \pi, z) = 0 \\ u(x, y, 0) = g(x, y), & 0 \leq x, y \leq \pi \end{cases}$$

We guess a solution of the form $u(x, y, z) = X(x)Y(y)Z(z)$ that solves the Laplace Equation. We follow the same procedure as with the Heat Equation in a Square. We get the $X(x) = \sin(nx)$, $Y(y) = \sin(my)$. Solving $Z(z)$ will yield a function of hyperbolic sine/cosine.

Idea: We solve the ODE $Z'' + c_1 Z = 0$. Since $c_1 = -(n^2 + m^2)$ (see Heat Eq in a Square), we have $Z'' = (n^2 + m^2)Z$. This gives

$$Z(z) = Ae^{\sqrt{n^2+m^2}z} + Be^{-\sqrt{n^2+m^2}z}$$

Plugging in the initial conditions $Z(\pi) = 0$

$$B = -Ae^{2\sqrt{m^2+n^2}\pi}$$

And so

$$Z(z) = 2Ae^{\sqrt{n^2+m^2}\pi}\left(\frac{e^{\sqrt{n^2+m^2}(z-\pi)} - e^{\sqrt{n^2+m^2}(\pi-z)}}{2}\right)$$

Therefore Z is a multiple of $\sinh(\pi - z)$ and the candidate solution for $u(x, y, z)$ is

$$u(x, y, z) \sim \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \alpha_{nm} \sinh\left(\sqrt{n^2+m^2}(\pi - z)\right)\sin(nx)\sin(my)$$

Plugging in our initial condition $u(x, y, 0) = g(x, y)$ yields

$$u(x, y, z) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} d_{nm}\frac{\sinh\left(\sqrt{n^2+m^2}(\pi - z)\right)}{\sinh\sqrt{n^2+m^2}\pi}\sin(nx)\sin(my)$$

Note that d_{nm} is obtained by setting it equal to $\alpha_{nm}\sinh\sqrt{n^2+m^2}\pi$, where

$$d_{nm} = \frac{1}{4\pi^2}\int_0^{\pi}\int_0^{\pi} g(x, y)\sin(nx)\sin(my)dxdy$$

Example: Solve the Laplace's Equation in a Cube

$$\begin{cases} \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} - \frac{\partial^2 u}{\partial z^2} = 0, & 0 < x, y, z < \pi \\ u(0, y, z) = u(\pi, y, z) = u(x, 0, z) = u(x, \pi, z) = 0 & \\ u(x, y, 0) = \sin(x)\sin^3(y), & 0 \le x, y \le \pi \end{cases}$$

Since $\sin^3(y) = \frac{3}{4}\sin(y) - \frac{1}{4}\sin(3y)$, we have that $d_{1,1} = \frac{3}{4}, d_{1,3} = -\frac{1}{4}$. We hence obtain the solution

$$u(x, y, z) = \frac{3\sinh(\sqrt{2}(\pi - z))}{4\sinh\sqrt{2}\pi}\sin(x)\sin(y) - \frac{\sinh(\sqrt{10}(\pi - z))}{4\sinh\sqrt{10}\pi}\sin(x)\sin(3y)$$

Example: Solve the Modified Laplace's Equation on a Cube

$$\begin{cases} \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} - \frac{\partial^2 u}{\partial z^2} - u = 0, & 0 < x < \pi, 0 < y < \frac{\pi}{2}, 0 < z < 1 \\ u(0,y,z) = u(x,0,z) = u(x,y,1) = 0 \\ \frac{\partial u}{\partial x}(x,y,z) = 0 \\ \frac{\partial u}{\partial y}\left(x, \frac{\pi}{2}, z\right) = 0 \\ u(x,y,0) = 2x - \pi \end{cases}$$

Guess the solution $u(x,y,z) = X(x)Y(y)Z(z)$. Then

$$X''YZ + XY''Z + XYZ'' - XYZ = 0 \Longrightarrow \frac{X''}{X} + \frac{Y''}{Y} + \frac{Z''}{Z} - 1 = c_1$$

$$\Longrightarrow \frac{X''}{X} + \frac{Y''}{Y} = -\frac{Z''}{Z} + c_1 \Longrightarrow \frac{X''}{X} = -\frac{Y''}{Y} + c_1 = c_2.$$

Solving the below ODE

$$\begin{cases} X'' = c_2 X \\ X(0) = X'(\pi) = 0 \end{cases}$$

Guess $X(x) = A\cos(\sqrt{c_2}x) + B\sin(\sqrt{c_2}x)$.

$$X(0) = 0 \Longrightarrow A = 0, X'(\pi) = 0 \Longrightarrow \cos(\sqrt{c_1}x) = 0 \Longrightarrow c_1 = -\left(\frac{2n-1}{2}\right)^2$$

So $X(x) = \sin\left(\left(\frac{2n-1}{2}\right)x\right)$.
Now we solve the ODE for $Y(y)$

$$\begin{cases} Y'' = (c_1 - c_2)Y \\ Y(0) = Y'\left(\frac{\pi}{2}\right) = 0 \end{cases}$$

Guess $Y(y) = A\cos\left(\sqrt{c_1 - c_2}y\right) + B\sin\left(\sqrt{c_1 - c_2}y\right)$. We follow the same procedure for $X(x)$, this time let $\sqrt{c_1 - c_2} = 2m - 1$. So $c_1 - c_2 = (2m-1)^2$.
Therefore $Y(y) = \sin((2m-1)y)$.
Also, $c_1 - c_2 = -(2m-1)^2 \Longrightarrow c_1 = -(2m-1) + c_2 = -(2m-1) - \left(\frac{2n-1}{2}\right)^2$.
Solving $Z(z)$

$$-\frac{Z''}{Z} + 1 = c_1 \Longrightarrow Z'' = (1 - c_1)Z$$

$$\begin{cases} Z'' = (c_1 - c_2)Z \\ Z(1) = 0 \end{cases}$$

We obtain

$$Z(z) = \sinh(\sqrt{1-c_1}) = \sinh\sqrt{1+\left(\frac{2n-1}{2}\right)^2+(2m-1)^2}(1-z)$$

Therefore the candidate solution for the series solution is

$$u(x,y,z) \sim \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} d_{nm}\sinh\left(\sqrt{1+\left(\frac{2n-1}{2}\right)^2+(2m-1)^2}(1-z)\right)\sin\left(\frac{2n-1}{2}x\right)\sin((2m-1)y)$$

Plugging in initial condition yields

$$\frac{\partial u}{\partial z}|_{z=0} = -\sqrt{1+\left(\frac{2n-1}{2}\right)^2+(2m-1)^2}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} d_{nm}\cosh\left(\sqrt{1+\left(\frac{2n-1}{2}\right)^2+(2m-1)^2}\right)$$

$$\sin\left(\frac{2n-1}{2}x\right)\sin((2m-1)y) = 2x-\pi$$

And so

$$d_{nm} = -\frac{\frac{4}{\pi^2}\int_0^{\frac{\pi}{2}}\int_0^{\pi}(2x-\pi)\sin\left(\frac{2n-1}{2}x\right)\sin((2m-1)y)dxdy}{\sqrt{1+\left(\frac{2n-1}{2}\right)^2+(2m-1)^2}\cosh\left(\sqrt{1+\left(\frac{2n-1}{2}\right)^2+(2m-1)^2}\right)}$$

We have

$$\int_0^{\frac{\pi}{2}}\int_0^{\pi}(2x-\pi)\sin\left(\frac{2n-1}{2}x\right)\sin((2m-1)y)dxdy = \frac{2\pi}{2m-1}\left(\frac{1}{2n-1}-\frac{2(-1)^n}{(2n-1)^2}\right)$$

By plugging in the above integral in the expression for d_{nm} we get our final solution

$$u(x,y,z) = \frac{8}{\pi}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{\left(\frac{1}{2n-1}-\frac{2(-1)^n}{(2n-1)^2}\right)\sinh\left(\sqrt{1+\left(\frac{2n-1}{2}\right)^2+(2m-1)^2}(1-z)\right)}{(2m-1)\sqrt{1+\left(\frac{2n-1}{2}\right)^2+(2m-1)^2}\cosh\left(\sqrt{1+\left(\frac{2n-1}{2}\right)^2+(2m-1)^2}\right)}$$

$$\cdot\sin\left(\frac{2n-1}{2}x\right)\sin((2m-1)y)$$

Laplace's Equation in a Cylinder

Consider the PDE

$$\begin{cases} \frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}+\frac{\partial^2 u}{\partial z^2}=0 & 0<r<R_1, -\pi\le\theta<\pi, 0<z<\pi \\ u(r,\theta,0)=u(r,\theta,\pi)=0 & 0<r<R_1, -\pi\le\theta<\pi \\ u(R_1,\theta,z)=g(\theta,z) & -\pi\le\theta\le\pi, 0\le z\le\pi \end{cases}$$

Let $R(r)\Theta(\theta)Z(z)$ be a function that satisfies the equation. Then, we have that

$$R''\Theta Z + \frac{1}{r}R\Theta Z + \frac{1}{r^2}R\Theta'' Z + R\Theta Z'' = 0$$

Dividing by $R\Theta Z$ yields

$$\frac{R'' + \frac{1}{r}R'}{R} + \frac{1}{r^2}\frac{\Theta''}{\Theta} = -\frac{Z''}{Z} = C_1 \text{ and } \frac{r^2R'' + rR'}{R} - C_1r^2 = -\frac{\Theta}{\theta} = C_2.$$

We get $Z(z) = \sin(nx)$ and $c_1 = n^2$ for $n \geq 1$. As for $\Theta(\theta)$, the solution to its ODE is $\Theta(\theta) = \sin(m\theta) + \cos(m\theta)$ for $C_2 = m^2$. Finding $R(r)$ involves solving a special type of ODE: the Modified Bessel's Equation. We will see that its corresponding equation for R is

$$r^2R'' + rR' - (n^2r^2 + m^2)R = 0$$

The components to solving the ODE is very technical, involving manipulation of power series. The final form of the solution is

$$I_m(\gamma) = \sum_{\ell=0}^{\infty} \frac{1}{\ell!(\ell+m)!}\left(\frac{\ell}{2}\right)^{m+2\ell}.$$

If we choose $R_{mn}(R_1) = 1$ for all m, n, then the solution for $R_{mn}(r)$ is

$$R_{mn}(r) = \frac{I_m(nr)}{I_m(nR_1)}$$

where $I_m(nr)$ and $I_m(nR_1)$ are the associated Bessel's equations.
Therefore, the final solution for the PDE becomes

$$\begin{aligned} u(r,\theta,z) = \frac{1}{2}\sum_{n=1}^{\infty} c_{n0}\frac{I_0(nr)}{I_0(nR_1)}\sin(nz) \\ + \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{I_0(nr)}{I_0(nR_1)}\sin(nz)(c_{nm}\cos(m\theta) + d_{nm}\sin(m\theta)) \end{aligned}$$

If we set $r = R_1$ then the Bessel's equations vanish and we get

$$g(\theta,z) = \frac{1}{2}\sum_{n=1}^{\infty} c_{n0}\sin(nz) + \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\sin(nz)(c_{nm}\cos(m\theta) + d_{nm}\sin(m\theta))$$

with the Fourier coefficients

$$c_{nm} = \frac{2}{\pi^2}\int_0^{\pi}\int_{-\pi}^{\pi} g(\theta,z)\sin(nz)\cos(m\theta)d\theta dz$$

$$d_{nm} = \frac{2}{\pi^2} \int_0^{\pi} \int_{-\pi}^{\pi} g(\theta, z) \sin(nz) \sin(m\theta) d\theta dz$$

Example: Solve Laplace's Equation in a Cylinder

$$\begin{cases} \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2} = 0 & 0 < r < 1, -\pi \leq \theta < \pi, 0 < z < \pi \\ u(r, \theta, 0) = u(r, \theta, \pi) = 0 & 0 < r < 1, -\pi \leq \theta < \pi \\ u(1, \theta, z) = z(\pi - z)\cos^2\theta & -\pi \leq \theta \leq \pi, 0 \leq z \leq \pi \end{cases}$$

First notice that $\cos^2\theta = \frac{1}{2} + \frac{1}{2}\cos(2\theta)$ and finding the sine series for $z(\pi - z)\sin(nz)$ yields

$$z(\pi - z) = \sum_{n=1}^{\infty} \frac{4(1 - (-1)^n)}{\pi n^3} \sin(nz)$$

d_{nm} has no terms because we only have cosine terms and the solution is defined for $m = 0, 2$ but is zero otherwise. So, the final solution is

$$u(r, \theta, z) = \sum_{n=1}^{\infty} \frac{2(1 - (-1)^n}{n^3\pi} \cdot \frac{I_0(nr)}{I_0(n)} \sin(nz) + \sum_{n=1}^{\infty} \frac{2(1 - (-1)^n)}{n^3\pi} \frac{I_2(nr)}{I_2(n)} \sin(nz)\cos(2\theta)$$

Damped Waves in a Square
Consider the PDE

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} + 2a\frac{\partial u}{\partial t} - c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = 0 & 0 < x, y < \pi, t > 0 \\ u(0,y,t) = u(\pi,y,t) = u(x,0,t) = u(x,\pi,t) = 0 & t \geq 0 \\ u(x,y,0) = f(x,y) & 0 \leq x, y \leq \pi \\ \frac{\partial u}{\partial t}(x,y,0) = 0 \end{cases}$$

Let $X(x)Y(y)T(t)$ be a function that satisfies the equation. Then

$$XYT'' + 2aXYT' - c^2(X''YT + XY''T) = 0$$

Dividing by $XYTc^2$ yields

$$\frac{T''}{c^2T} + \frac{2aT'}{c^2T} - \frac{X''}{X} - \frac{Y''}{Y} = 0 \Longrightarrow \frac{T'' + 2aT'}{c^2T} - \frac{Y''}{Y} = \frac{X''}{X} = c_1$$

Standard arguments lend $X(x)$ to the solution $X(x) = \sin(nx)$ for $c_1 = -n^2$. Now we find a solution for $Y(y)$.

$$\frac{T'' + 2aT}{c^2T} - c_1 = \frac{Y''}{Y} = c_2$$

By similar reasoning, $Y(y) = \sin(my)$ for $c_2 = -m^2$. For $T(t)$, we get a characteristic ODE of the form

$$T^2 + 2aT + c^2(n^2 + m^2) = 0$$

The corresponding solution depends on the discriminant for given a, c, n, m. We have that

$$T_{nm}(t) = e^{-at}\left[\cosh\sqrt{a^2 - c^2(n^2+m^2)}t + \frac{a}{\sqrt{a^2 - c^2(n^2+m^2)}}\sinh\sqrt{a^2 - c^2(n^2+m^2)}t\right]$$
$$\text{for } \sqrt{n^2+m^2} < \frac{a}{c}$$

$$T_{nm}(t) = e^{-at}(1 + at) \text{ for } \sqrt{n^2+m^2} = \frac{a}{c}$$

$$T_{nm}(t) = e^{-at}\left[\cos\sqrt{c^2(n^2+m^2) - a^2}t + \frac{a}{\sqrt{c^2(n^2+m^2) - a^2}}\sin\sqrt{c^2(n^2+m^2) - a^2}t\right]$$
$$\text{for } \sqrt{n^2+m^2} > \frac{a}{c}$$

Hence we obtain the solution

$$u(x,y,t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} d_{nm} T_{nm}(t)\sin(nx)\sin(my)$$

Where d_{nm} is the Fourier coefficients of the double sine-series from the condition $u(x,y,0) = f(x,y)$.

$$d_{nm} = \frac{4}{\pi^2}\int_0^\pi\int_0^\pi f(x,y)\sin(nx)\sin(my)dxdy$$

Example: Solve the Damped Wave equation in a Square

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} + 4\frac{\partial u}{\partial t} - \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = 0 & 0 < x, y < \pi, t > 0 \\ u(0,y,t) = u(\pi,y,t) = u(x,0,t) = u(x,\pi,t) = 0 & t \geq 0 \\ u(x,y,0) = \sin^3 x\sin^3 y & 0 \leq x, y \leq \pi \\ \frac{\partial u}{\partial t}(x,y,0) = 0 \end{cases}$$

Here $a = 2, c = 1$. By decomposition of $\sin^3 x\sin^3 y$

$$\sin^3 x\sin^3 y = \frac{9}{16}\sin x\sin y - \frac{3}{16}\sin x\sin(3y) - \frac{3}{16}\sin(3x)\sin y + \frac{1}{16}\sin(3x)\sin(3y)$$

Finding each $T_{mn}(t)$ is simply checking the conditions of the discriminant. For example, $T_{1,1}(t)$ has $\sqrt{2} < 2$, so we use the first equation in the last page. Similar arguments yield to the final solution

$$u(x,y,t) = \frac{9}{16}e^{-2t}[\cosh\sqrt{2}t + \sqrt{2}\sinh\sqrt{2}t]\sin x\sin y - \frac{3}{16}e^{-2t}\left[\cos\sqrt{6}t + \frac{2}{\sqrt{6}}\sin\sqrt{6}t\right]\sin x\sin(3y)$$
$$-\frac{3}{16}e^{-2t}\left[\cos\sqrt{6}t + \frac{2}{\sqrt{6}}\sin\sqrt{6}t\right]\sin(3x)\sin y + \frac{1}{16}e^{-2t}\left[\cos\sqrt{14}t + \frac{2}{\sqrt{14}}\sin\sqrt{14}t\right]\sin(3x)\sin(3y)$$

6 About Uniform Convergence and Differentiation

When we find a series solution to a PDE, we require uniform convergence and term-by-term differentiation to know that the series converges to the solution everywhere. To prove that a series is uniformly convergent, we show that the series solution is absolutely convergent. To prove that a series can be differentiated term by term, show that the derivative of a series converges absolutely.

Example: Let the following series be a solution to a PDE. Show that it converges uniformly.

$$u(x,y) = \sum_{k=1}^{\infty} b_{2k-1}(y)\sin((2k-1)x)$$

where

$$\begin{aligned} b_{2k-1} = \frac{8}{(2k-1)^5\pi}y^2 - \frac{8}{(2k-1)^5}y + \frac{16}{(2k-1)^7\pi} \\ + \frac{16}{(2k-1)^7\pi} \cdot \frac{\sinh((2k-1)(y-\pi)) - \sinh((2k-1)y)}{\sinh((2k-1)\pi)} \end{aligned}$$

First, we have $|\sin((2k-1)x)| \leq 1$. Now we show $b_{2k-1}(y)$ converges. We rewrite the sinh terms as

$$\begin{aligned} \sinh((2k-1)(y-\pi)) &= e^{(2k-1)(y-\pi)} - e^{(2k-1)(\pi-y)} \\ \sinh((2k-1)y) &= e^{(2k-1)y} - e^{-(2k-1)y} \\ \sinh((2k-1)\pi) &= e^{(2k-1)\pi} - e^{-(2k-1)\pi} \end{aligned}$$

so $\frac{\sinh((2k-1)(y-\pi)) - \sinh((2k-1)y)}{\sinh((2k-1)\pi)}$ becomes

$$\frac{e^{(2k-1)(y-\pi)} - e^{(2k-1)(\pi-y)} - e^{(2k-1)y} - e^{-(2k-1)y}}{e^{(2k-1)\pi} - e^{-(2k-1)\pi}}.$$

We factor out $e^{(2k-1)\pi}$ from both the numerator and the denominator to obtain

$$\frac{e^{(2k-1)\pi}\left(e^{(2k-1)(y-2\pi)} - e^{-(2k-1)(y-\pi)}\right)}{e^{(2k-1)\pi}\left(1 - e^{-2(2k-1)\pi}\right)} = \frac{e^{(2k-1)(y-2\pi)} - e^{-(2k-1)(y-\pi)}}{1 - e^{-2(2k-1)\pi}}$$

We can verify that on $y \in [0,\pi]$, the numerator is always smaller than 1, and the denominator is always larger than $1 - e^{-2\pi}$. It should be obvious that

$$\frac{1}{1-e^{-2\pi}} < 2.$$

Now we look at the other terms. We have that

$$b_{2k-1} \leq \left|\frac{8\pi + 8\pi}{(2k-1)^5}\right| + \left|\frac{16}{(2k-1)^7\pi}\right| + 2 \cdot \left|\frac{16}{(2k-1)^7\pi}\right|$$

Since $\frac{1}{(2k-1)^7} \leq \frac{1}{(2k-1)^5}$ we finally have

$$b_{2k-1}(y) \leq \frac{16}{(2k-1)^5} + \frac{48}{(2k-1)^5} = \frac{16\pi + 48}{(2k-1)^5}$$

which converges by $p-$series. So, $b_{2k-1}(y)$ converges by comparison to the above series, and we have uniform converge.

The proof for term-by-term differentiation is the same idea. This time, we show that $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 u}{\partial y^2}$ are all uniformly convergent. Proving all of these follows the process we did for the actual series solution. Therefore, term-by-term differentiation is valid.

7 Appendix and Other Resources

Product-to-Sum Identities

$$\cos(a)\cos(b) = \frac{1}{2}(\cos(a+b) + \cos(a-b))$$

$$\sin(a)\sin(b) = \frac{1}{2}(\cos(a-b) - \cos(a+b))$$

$$\cos(a)\sin(b) = \frac{1}{2}(\sin(a+b) + \sin(a-b))$$

$$\sin(a)\cos(b) = \frac{1}{2}(\sin(a+b) - \sin(a-b))$$

Deriving $\sin^5 x$: We use the fact that $\sin^3 x = \frac{3}{4}\sin x - \frac{1}{4}\sin(3x)$ to lower the computation.

$$\sin^5 x = \sin^3 x \cdot \sin^2 x = \left(\frac{3}{4}\sin x - \frac{1}{4}\sin(3x)\right)\left(\frac{1}{2}(1 - \cos(2x))\right)$$

$$\sin x \cos(2x) = \frac{1}{2}(\sin(3x) - \sin x), \sin(3x)\cos(2x) = \frac{1}{2}(\sin(5x) + \sin(x))$$

So

$$\begin{aligned}\sin^5 x &= \frac{3}{8}\sin(x) - \frac{1}{8}\sin(3x) - \frac{3}{16}\sin(3x) + \frac{3}{16}\sin(x) + \frac{1}{16}\sin(5x) + \frac{1}{16}\sin(x) \\ &= \frac{5}{8}\sin x - \frac{5}{16}\sin(3x) + \frac{1}{16}\sin(5x)\end{aligned}$$